

# A Central Limit Theorem for averaged stochastic gradient algorithms in Hilbert spaces and online estimation of the asymptotic variance. Application to the Geometric Median and Quantiles

Antoine Godichon-Baggioni  
 Institut de Mathématiques de Toulouse,  
 INSA de Toulouse, 31400 Toulouse, France  
 email: godichon@insa-toulouse.fr

February 6, 2017

## Abstract

Stochastic gradient algorithms are more and more studied since they can deal efficiently and online with large samples in high dimensional spaces. In this paper, we first establish a Central Limit Theorem for these estimates as well as for their averaged version in general Hilbert spaces. Moreover, since having the asymptotic normality of estimates is often unusable without an estimation of the asymptotic variance, we introduce a recursive algorithm of the asymptotic variance of the averaged estimator, and we establish its almost sure rate of convergence as well as its rate of convergence in quadratic mean. Finally, an example in robust statistics is given: the estimation of Geometric Quantiles and of the Geometric Median.

**Keywords:** Stochastic Gradient Algorithm, Averaging, Central Limit Theorem, Asymptotic Variance, Geometric Median, Geometric Quantiles.

## 1 Introduction

High Dimensional and Functional Data Analysis are interesting domains which do not have stopped growing for many years. To consider these kinds of data, it is more and more important to think about methods which take into account the high dimension as well as the possibility of having large samples. In this paper, we focus on an usual stochastic optimization problem which consists in estimating

$$m := \arg \min_{h \in H} \mathbb{E} [g(X, h)],$$

where  $X$  is a random variable taking values in a space  $\mathcal{X}$  and  $g : \mathcal{X} \times H \rightarrow \mathbb{R}$ , where  $H$  is a separable Hilbert space. In order to build an estimator of  $m$ , an usual method was

to consider the solver of the problem generated by the sample, i.e to consider  $M$ -estimates (see [Huber and Ronchetti \(2009\)](#) and [Maronna et al. \(2006\)](#) among others). In order to build these estimates, deterministic convex optimization algorithms (see [Boyd and Vandenberghe \(2004\)](#)) are often used (see [Vardi and Zhang \(2000\)](#), [Oja and Niinimaa \(1985\)](#) in the case of the median), and these methods are really efficient in small dimensional spaces.

Nevertheless, in a context of high dimensional spaces, this kind of method can encounter many computational problems. The main ones are that it needs to store all the data, which can be expensive in term of memory and that they cannot deal online with the data. In order to overcome this, stochastic gradient algorithms ([Robbins and Monro \(1951\)](#)) are efficient candidates since they do not need to store the data into memory, and they can be easily updated, which is crucial if the data arrive sequentially (see [Duflo \(1996\)](#), [Duflo \(1997\)](#), [Kushner and Yin \(2003\)](#) or [Nemirovski et al. \(2009\)](#) among others). In order to improve the convergence, [Ruppert \(1988\)](#) and [Polyak and Juditsky \(1992\)](#) introduced its averaged version (see also [Dippon and Renz \(1997\)](#) for the weighted version). These algorithms have become crucial to statistics and modern machine learning ([Bach and Moulines \(2013\)](#), [Bach \(2014\)](#), [Juditsky et al. \(2014\)](#)). There are already many results on these algorithms in the literature, that we can split into two parts: asymptotic results, such as almost sure rates of convergence (see for instance [Duflo \(1997\)](#), [Pelletier \(1998\)](#), [Mokkadem and Pelletier \(2006\)](#)) and non asymptotic ones, such as quadratic error bounds (see [Carlot et al. \(2015\)](#), [Godichon-Baggioni \(2016a\)](#), [Bach and Moulines \(2013\)](#)). Nevertheless, the framework proposed to get non asymptotic results in [Bach and Moulines \(2013\)](#) forces the function we would like to minimize on being uniformly strongly convex. In order to obtain asymptotic results, [Pelletier \(1998\)](#) and [Pelletier \(2000\)](#) give a general framework for which no strongly convexity assumption is needed, but the results are only proven in finite dimensional spaces.

In a recent work, [Godichon-Baggioni \(2016b\)](#) introduce a new framework, with only locally strongly convexity assumptions (and non global ones), in general Hilbert spaces, which allows to obtain almost sure rates and  $L^p$  rates of convergence. In keeping with it, and in order to have a deeper study of the stochastic gradient algorithm as well as of its averaged version (up to a new assumption), we first give the asymptotic normality of the estimates. In a second time, since a Central Limit Theorem is often unusable without an estimation of the variance, we introduce a recursive algorithm, inspired by [Gahbiche and Pelletier \(2000\)](#), which enables us to estimate the asymptotic variance of the averaged estimator, and we establish the rates of convergence of these estimates. As far as we know, there was no efficient and recursive estimate of the asymptotic variance in the literature. Finally, we will give two applications in robust statistics: the geometric median (see [Haldane \(1948\)](#) or [Kemperman \(1987\)](#)) and the geometric quantiles (see [Chaudhuri \(1996\)](#) and [Chakraborty and Chaudhuri \(2014\)](#)), which are useful robust indicators in statistics. Indeed, they are often used in data depth and outliers detection ([Serfling \(2006\)](#), [Hallin and Paindaveine \(2006\)](#)), as well as for robust estimation of the mean and variance (see [Minsker et al. \(2014\)](#)), or for Robust Principal Component Analysis ([Gervini \(2008\)](#), [Kraus and Panaretos \(2012\)](#)),

Cardot and Godichon-Baggioni (2015)).

The paper is organized as follows: Section 2 recalls the framework introduced by Godichon-Baggioni (2016b) before giving two new assumptions which allow to get the rate of convergence of the estimators of the asymptotic variance. In section 3, the stochastic gradient algorithm as well as its averaged version are introduced and their asymptotic normality are given. The recursive estimator of the asymptotic variance is given in Section 4 and its almost sure as well as its quadratic mean rates of convergence are established. An application on the recursive estimations of the geometric median and of the geometric quantiles as well as a short simulation study are given in Section 5. Finally, the proofs are postponed in Section 6 and in an Appendix.

## 2 Assumptions

Let  $H$  be a separable Hilbert space such as  $\mathbb{R}^d$  or  $L^2(I)$  (for some closed interval  $I \subset \mathbb{R}$ ), we denote by  $\langle \cdot, \cdot \rangle$  its inner product and by  $\|\cdot\|$  the associated norm. Let  $X$  be a random variable taking values in a space  $\mathcal{X}$ , and let  $G : H \rightarrow \mathbb{R}$  be the function we would like to minimize, defined for all  $h \in H$  by

$$G(h) := \mathbb{E} [g(X, h)], \quad (1)$$

where  $g : \mathcal{X} \times H \rightarrow \mathbb{R}$ . Moreover, let us suppose that the functional  $G$  is convex. Finally, let us introduce the space of linear operators on  $H$ , denoted by  $\mathcal{S}(H)$ , equipped with the Frobenius (or Hilbert-Schmidt) inner product, which is defined by

$$\langle A, B \rangle_F := \sum_{i \in I} \langle A(e_i), B(e_i) \rangle, \quad \forall A, B \in \mathcal{S}(H),$$

where  $(e_i)_{i \in I}$  is an orthonormal basis of  $H$ . We denote by  $\|\cdot\|_F$  the associated norm, and  $\mathcal{S}(H)$  is then a separable Hilbert space. Let us recall the framework introduced by Godichon-Baggioni (2016b):

- (A1) The functional  $g$  is Frechet-differentiable for the second variable almost everywhere. Moreover,  $G$  is differentiable and denoting by  $\Phi$  its gradient, there exists  $m \in H$  such that

$$\Phi(m) := \nabla G(m) = 0.$$

- (A2) The functional  $G$  is twice continuously differentiable almost everywhere and for all positive constant  $A$ , there is a positive constant  $C_A$  such that for all  $h \in \mathcal{B}(m, A)$ ,

$$\|\Gamma_h\|_{op} \leq C_A,$$

where  $\Gamma_h$  is the Hessian of the functional  $G$  at  $h$  and  $\|\cdot\|_{op}$  is the usual spectral norm for linear operators.

**(A3)** There exists a positive constant  $\epsilon$  such that for all  $h \in \mathcal{B}(m, \epsilon)$ , there is an orthonormal basis of  $H$  composed of eigenvectors of  $\Gamma_h$ . Moreover, let us denote by  $\lambda_{\min}$  the limit inf of the eigenvalues of  $\Gamma_m$ , then  $\lambda_{\min}$  is positive. Finally, for all  $h \in \mathcal{B}(m, \epsilon)$ , and for all eigenvalue  $\lambda_h$  of  $\Gamma_h$ , we have  $\lambda_h \geq \frac{\lambda_{\min}}{2} > 0$ .

**(A4)** There are positive constants  $\epsilon, C_\epsilon$  such that for all  $h \in \mathcal{B}(m, \epsilon)$ ,

$$\|\Phi(h) - \Gamma_m(h - m)\| \leq C_\epsilon \|h - m\|^2.$$

**(A5)** (a) There is a positive constant  $L_1$  such that for all  $h \in H$ ,

$$\mathbb{E} \left[ \|\nabla_h g(X, h)\|^2 \right] \leq L_1 \left( 1 + \|h - m\|^2 \right).$$

(a') There is a positive constant  $L_2$  such that for all  $h \in H$ ,

$$\mathbb{E} \left[ \|\nabla_h g(X, h)\|^4 \right] \leq L_2 \left( 1 + \|h - m\|^4 \right).$$

(b) For all integer  $q$ , there is a positive constant  $L_q$  such that for all  $h \in H$ ,

$$\mathbb{E} \left[ \|\nabla_h g(X, h)\|^{2q} \right] \leq L_q \left( 1 + \|h - m\|^{2q} \right).$$

Let us now introduce two new assumptions.

**(A6)** Let  $\varphi : H \longrightarrow \mathcal{S}(H)$  be the functional defined for all  $h \in H$  by

$$\varphi(h) := \mathbb{E} [\nabla_h g(X, h) \otimes \nabla_h g(X, h)].$$

(a) The functional  $\varphi$  is continuous at  $m$  with respect to the Frobenius norm:

$$\lim_{h \rightarrow m} \|\mathbb{E} [\nabla_h g(X, m) \otimes \nabla_h g(X, m)] - \mathbb{E} [\nabla_h g(X, h) \otimes \nabla_h g(X, h)]\|_F = 0.$$

(b) The functional  $\varphi$  is locally lipschitz on a neighborhood of  $m$ : there are positive constants  $\epsilon, C_\epsilon$ , such that for all  $h \in \mathcal{B}(m, \epsilon)$ ,

$$\|\mathbb{E} [\nabla_h g(X, m) \otimes \nabla_h g(X, m) - \nabla_h g(X, h) \otimes \nabla_h g(X, h)]\|_F \leq C_\epsilon \|h - m\|.$$

Note that assumptions **(A1)** to **(A5b)** are discussed in [Godichon-Baggioni \(2016b\)](#). Assumption **(A6b)** can be verified by giving a bound, on a neighborhood of  $m$ , of the derivative of the functional  $\varphi$ . This last assumption allows to give the rate of convergence of the estimators of the asymptotic variance.

**Remark 2.1.** Let  $h, h' \in H$ , the linear operator  $h \otimes h' : H \longrightarrow H$  is defined for all  $h'' \in H$  by  $h \otimes h'(h'') := \langle h, h'' \rangle h'$ . Moreover, note that

$$\|h \otimes h'\|_F = \|h\| \|h'\|. \quad (2)$$

### 3 The stochastic gradient algorithm and its averaged version

#### 3.1 The Robbins-Monro algorithm

In what follows, let  $X_1, \dots, X_n$  be independent random variables with the same law as  $X$ . The stochastic gradient algorithm is defined recursively for all  $n \geq 1$  by

$$m_{n+1} = m_n - \gamma_n \nabla_h g(X_{n+1}, m_n), \quad (3)$$

with  $m_1$  bounded and  $(\gamma_n)$  is a step sequence of the form  $\gamma_n := c_\gamma n^{-\alpha}$ , with  $c_\gamma > 0$  and  $\alpha \in (\frac{1}{2}, 1)$ . Moreover, let  $(\mathcal{F}_n)_{n \geq 1}$  be the sequence of  $\sigma$ -algebras defined for all  $n \geq 1$  by  $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$ . Then, the algorithm can be considered as a noisy (or stochastic) gradient algorithm since it can be written as

$$m_{n+1} = m_n - \gamma_n \Phi(m_n) + \gamma_n \xi_{n+1}, \quad (4)$$

where  $(\xi_n)$ , defined for all  $n \geq 1$  by  $\xi_{n+1} := \Phi(m_n) - \nabla_h g(X_{n+1}, m_n)$ , is a martingale differences sequence adapted to the filtration  $(\mathcal{F}_n)$ . Finally, note that under assumptions **(A1)** to **(A5a)**, it was proven in [Godichon-Baggioni \(2016b\)](#) that for all positive constant  $\delta$ ,

$$\|m_n - m\|^2 = o\left(\frac{(\ln n)^\delta}{n^\alpha}\right) \quad a.s. \quad (5)$$

Moreover, assuming also that **(A5b)** is fulfilled, for all positive integer  $p$ , there is a constant  $C_p$  such that for all  $n \geq 1$ ,

$$\mathbb{E} \left[ \|m_n - m\|^{2p} \right] \leq \frac{C_p}{n^{p\alpha}}. \quad (6)$$

In order to get a deeper study of this estimate, we now give its asymptotic normality.

**Theorem 3.1.** *Suppose assumptions **(A1)** to **(A5a')** and **(A6a)** hold. Then, we have the convergence in law*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{\gamma_n}} (m_n - m) \sim \mathcal{N}(0, \Sigma_{RM}),$$

with

$$\Sigma_{RM} := \int_0^{+\infty} e^{-s\Gamma_m} \Sigma' e^{-s\Gamma_m} ds, \quad \text{and} \quad \Sigma' := \mathbb{E} [\nabla_h g(X, m) \otimes \nabla_h g(X, m)].$$

**Remark 3.1.** *Note that an analogous result is given by [Pelletier \(1998\)](#) in the particular case of finite dimensional spaces. Nevertheless, the proof of this result cannot be directly applied in the infinite case. For example, it remains on the fact that the Hessian admits finite dimensional eigenspaces, or on the fact that the trace of a matrix is well defined.*

**Remark 3.2.** *Note that taking a step sequence of the form  $\gamma_n = \frac{c}{n}$  with  $c > \frac{1}{2\lambda_{\min}}$  is possible, and one can obtain the following asymptotic normality (see [Pelletier \(2000\)](#) for the case of finite dimensional spaces)*

$$\lim_{n \rightarrow \infty} \sqrt{n} (m_n - m) \sim \mathcal{N}(0, c\Sigma').$$

Nevertheless, knowing the conditions on  $c$  needs to have some information on the Hessian  $\Gamma_m$ , which can be as difficult as to estimate  $m$ . Moreover,  $c\Sigma'$  is not the optimal covariance (see [Duflo \(1997\)](#) and [Pelletier \(2000\)](#) for instance).

### 3.2 The averaged algorithm

As mentioned in Remark 3.2, having the parametric rate of convergence ( $O(\frac{1}{n})$ ) with the Robbins-Monro algorithm is possible taking a good choice of step sequence  $(\gamma_n)$ . Nevertheless, this choice is often complicated and the asymptotic variance which is obtained is not optimal. Then, in order to improve the convergence, let us now introduce the averaged algorithm (see [Ruppert \(1988\)](#) and [Polyak and Juditsky \(1992\)](#)) defined for all  $n \geq 1$  by

$$\bar{m}_n = \frac{1}{n} \sum_{k=1}^n m_k.$$

This can be written recursively for all  $n \geq 1$  as

$$\bar{m}_{n+1} = \bar{m}_n + \frac{1}{n+1} (m_{n+1} - \bar{m}_n). \quad (7)$$

It was proven in [Godichon-Baggioni \(2016b\)](#) that under assumptions **(A1)** to **(A5a)**, for all  $\delta > 0$ ,

$$\|\bar{m}_n - m\|^2 = o\left(\frac{(\ln n)^{1+\delta}}{n}\right) \quad a.s. \quad (8)$$

Suppose assumption **(A5b)** is also fulfilled, for all positive integer  $p$ , there is a positive constant  $C'_p$  such that for all  $n \geq 1$ ,

$$\mathbb{E} \left[ \|\bar{m}_n - m\|^{2p} \right] \leq \frac{C'_p}{n^p}. \quad (9)$$

Finally, in order to have a deeper study of this estimate, we now give its asymptotic normality.

**Theorem 3.2.** *Suppose assumptions **(A1)** to **(A5a')** and **(A6a)** are verified. Then, we have the convergence in law*

$$\lim_{n \rightarrow \infty} \sqrt{n} (\bar{m}_n - m) \sim \mathcal{N}(0, \Sigma),$$

with  $\Sigma := \Gamma_m^{-1} \Sigma' \Gamma_m^{-1}$ , and  $\Sigma' := \mathbb{E} [\nabla_h g(X, m) \otimes \nabla_h g(X, m)]$ .

**Remark 3.3.** *Note that as for the case of the Robbins-Monro algorithm, this result exists in the particular case of finite dimensional spaces in [Pelletier \(2000\)](#), but the proof can not be directly applied in general Hilbert spaces.*

## 4 Recursive estimation of the asymptotic variance

### 4.1 Some existing estimators

A first naive method to estimate the asymptotic variance could be to estimate the Hessian  $\Gamma_m$  and the variance  $\Sigma'$  as follows

$$\begin{aligned}\Gamma_m^{(n+1)} &= \Gamma_m^{(n)} + \frac{1}{n+1} \left( \nabla_h^2 g(X_{n+1}, \bar{m}_n) - \Gamma_m^{(n)} \right), \\ \Sigma'_{n+1} &= \Gamma_m^{(n)} + \frac{1}{n+1} \left( \nabla_h g(X_{n+1}, \bar{m}_n) \otimes \nabla_h g(X_{n+1}, \bar{m}_n) - \Sigma'_n \right),\end{aligned}$$

but the main problem is that under assumptions **(A2)**, **(A3)** and **(A5)**, if  $H$  is an infinite dimensional spaces, then

$$\|\Gamma_m\|_F = \infty, \quad \text{while} \quad \left\| \Gamma_m^{-1} \Sigma' \Gamma_m^{-1} \right\|_F \leq \frac{\sqrt{L_1}}{\lambda_{\min}}.$$

An other problem is that, in order to get a recursive estimator of the asymptotic variance, it needs to invert a matrix at each iteration, which costs much calculus time in high dimensional spaces. A second estimator of the asymptotic variance was introduced in [Pelletier \(2000\)](#). It is defined for all  $n \geq 1$  by

$$\hat{\Sigma}_n = \frac{1}{\ln n} \sum_{k=1}^n (m_k - \bar{m}_n) \otimes (m_k - \bar{m}_n). \quad (10)$$

Nevertheless, suppose assumptions **(A1)** to **(A6b)** hold, then

$$\mathbb{E} \left[ \left\| \hat{\Sigma}_n - \Sigma \right\|_F^2 \right] = O \left( \frac{1}{\ln n} \right).$$

Thus, this estimator faces two main problems: it is not recursive and it converges very slowly. Finally, in order to solve the second problem, a faster algorithm was introduced by [Gahbiche and Pelletier \(2000\)](#), defined for all  $n \geq 1$  by

$$\tilde{\Sigma}_n := \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^\delta k^s} \exp \left( -\frac{k^{1-s}}{1-s} \right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - \bar{m}_n) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - \bar{m}_n) \right), \quad (11)$$

with  $(1+\alpha)/2 < s < 1$  and  $s/2 < \delta < (1+s)/2$ . In the case of finite dimensional spaces, the following convergence in probability is given (under some assumptions)

$$\frac{n^{1/2-s/2}}{(\ln \ln n)^{-c}} \left\| \tilde{\Sigma}_n - \Sigma \right\|_{op} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

with  $c > 0$ . A first technical problem is that only the convergence in probability is given, in the case of finite dimensional spaces, and for the usual spectral norm. A second one is that it is not recursive and it cannot be easily updated.

## 4.2 A recursive and fast estimate

We now give a recursive version of the algorithm defined by (11) to estimate the asymptotic variance in separable Hilbert spaces, before establishing its rates of convergence (almost sure and in quadratic mean). This algorithm is defined by

$$\Sigma_n := \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - \bar{m}_j) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - \bar{m}_j) \right), \quad (12)$$

with

$$(1+\alpha)/2 < s < 1, \quad \text{and} \quad s/2 < \delta < (1+s)/2. \quad (13)$$

This can be written recursively for all  $n \geq 1$  as

$$\begin{aligned} V_{n+1} &= V_n + \exp\left(\frac{(n+1)^{1-s}}{2(1-s)}\right) (m_{n+1} - \bar{m}_{n+1}), \\ \Sigma_{n+1} &= \left(\frac{n}{n+1}\right)^{1-\delta} \Sigma_n + \frac{1-\delta}{(n+1)^{\delta+s}} \exp\left(-\frac{(n+1)^{1-s}}{1-s}\right) V_{n+1} \otimes V_{n+1}, \end{aligned}$$

with  $V_1 = \Sigma_1 = 0$ . Then, contrary to previous algorithms, this one does not need to store all the estimations into memory and can be easily updated, which is crucial when the data arrive sequentially. Finally, the following theorem ensures that it is quite fast.

**Theorem 4.1.** *Suppose assumptions (A1) to (A5a') and (A6b) hold. Then, the sequence  $(\Sigma_n)$  defined by (12) verifies for all positive constant  $\gamma$ ,*

$$\|\Sigma_n - \Sigma\|_F^2 = o\left(\frac{(\ln n)^\gamma}{n^{1-s}}\right) \quad a.s.$$

Moreover, suppose (A5b) holds too, there is a positive constant  $C$  such that for all  $n \geq 1$ ,

$$\mathbb{E} \left[ \|\Sigma_n - \Sigma\|_F^2 \right] \leq \frac{C}{n^{1-s}}$$

The proof is given in Section 6.

**Corollary 4.1.** *Suppose assumptions (A1) to (A5a') and (A6b) hold. Then, for all positive constant  $\gamma$ ,*

$$\|\tilde{\Sigma}_n - \Sigma\|_F^2 = o\left(\frac{(\ln n)^\gamma}{n^{1-s}}\right) \quad a.s.$$

Moreover, suppose (A5b) holds too, there is a positive constant  $C$  such that for all  $n \geq 1$ ,

$$\mathbb{E} \left[ \|\tilde{\Sigma}_n - \Sigma\|_F^2 \right] \leq \frac{C}{n^{1-s}}$$

**Remark 4.1.** *Operator  $\Sigma$  is self-adjoint and non negative, so that it admits a spectral decomposition  $\Sigma = \sum_{i \in I} \eta_i v_i \otimes v_i$ , where  $\eta_1 \geq \eta_2 \geq \dots \geq 0$  is the sequence of ordered eigenvalues associated to the orthonormal eigenvectors  $v_j, j \in I$ . Using the Karhunen-Loève's expansion of  $\Sigma$ , we directly get*



that

$$\|\Sigma\|^2 = \sum_{j \in I} \eta_j^2 V_j^2$$

where  $V_1, V_2, \dots$  are i.i.d. centered Gaussian variables with unit variance. Thus, the distribution of  $\|V\|^2$  is a mixture of independent Chi-square random variables with one degree of freedom. Then it is interesting to estimate the main eigenvalues to build precise confidence balls. Let  $\eta_1 > \eta_2 > \dots > \eta_q$  be the  $q$  main eigenvalues, it is then possible to estimate recursively  $\eta_1, \dots, \eta_q$  as well as the associated normalized eigenvectors  $u_1, \dots, u_q$  as follows:

$$v_{j,n+1} = v_{j,n} + \frac{1}{n+1} \left( \bar{\Sigma}_{n+1} \frac{v_{j,n}}{\|v_{j,n}\|} - v_{j,n} \right), \quad j = 1, \dots, q \quad (14)$$

combined with an orthogonalization by deflation of  $v_{1,n+1}, \dots, v_{q,n+1}$ . This recursive algorithm is based on ideas developed by [Weng et al. \(2003\)](#) that are related to the power method for extracting eigenvectors. The estimated eigenvectors  $v_{1,n+1}, \dots, v_{q,n+1}$  tend (up to the sign change) to  $\eta_1 u_1, \dots, \eta_q u_q$ .

## 5 Application to the geometric median and geometric quantiles

### 5.1 Definition and results

Let  $H$  be a separable Hilbert space and let  $X$  be a random variable taking values in  $H$ . Let  $v \in H$  such that  $\|v\| < 1$ , the geometric quantile  $m^v$  corresponding to the direction  $v$  (see [Chaudhuri \(1996\)](#)) is defined by

$$m^v := \arg \min_{h \in H} \mathbb{E} [\|X - h\| - \|X\|] - \langle h, v \rangle, \quad (15)$$

and in a particular case, the geometric median  $m$  (see [Haldane \(1948\)](#)) corresponds to the case where  $v = 0$ . We consider from now that the following usual assumptions are fulfilled ([Cardot et al. \(2013\)](#)):

**(H1)** The random variable  $X$  is not concentrated on a straight line: for all  $h \in H$ , there is  $h' \in H$  such that  $\langle h, h' \rangle = 0$  and

$$\text{Var} (\langle X, h' \rangle) > 0.$$

**(H2)** The random variable  $X$  is not concentrated around single points: for all positive constant  $A$ , there is a positive constant  $C_A$  such that for all  $h \in \mathcal{B}(0, A)$ ,

$$\mathbb{E} \left[ \frac{1}{\|X - h\|} \right] \leq C_A, \quad \mathbb{E} \left[ \frac{1}{\|X - h\|^2} \right] \leq C_A.$$

Let us now denote by  $G_v : H \longrightarrow \mathbb{R}$  and  $g_v : H \times H$  the functions defined for all  $h, x \in H$  by

$$G_v(h) := \mathbb{E} [\|X - h\| - \|X\|] - \langle h, v \rangle, \quad g_v(x, h) := \|x - h\| - \|x\| - \langle h, v \rangle.$$

Under **(H1)** and **(H2)**, the functional  $G_v$  is twice Fréchet-differentiable and it was proven (see [Kemperman \(1987\)](#), [Cardot et al. \(2013\)](#) and [Godichon-Baggioni \(2016b\)](#)) that assumptions **(A1)** to **(A5b)** are verified. Finally, the following lemma ensures that Assumptions **(A6a)** and **(A6b)** are also fulfilled.

**Lemma 5.1.** *Suppose assumption **(H2)** holds. Then, there are positive constants  $\epsilon, C_\epsilon$  such that for all  $h \in \mathcal{B}(m^v, \epsilon)$ ,*

$$\|\mathbb{E} [\nabla_h g_v(X, m^v) \otimes \nabla_h g_v(X, m^v) - \nabla_h g_v(X, h) \otimes \nabla_h g_v(X, h)]\|_F \leq C_\epsilon \|h - m^v\|.$$

The proof is given in an Appendix. Then, it is possible to estimate simultaneously and recursively the geometric quantile  $m^v$  as well as the asymptotic variance of the averaged estimator as follows:

$$\begin{aligned} m_{n+1}^v &= m_n^v + \gamma_n \left( \frac{X_{n+1} - m_n^v}{\|X_{n+1} - m_n^v\|} + v \right), \\ \bar{m}_{n+1}^v &= \bar{m}_n^v + \frac{1}{n+1} (m_{n+1}^v - \bar{m}_n^v), \\ V_{n+1}^v &= V_n^v + \exp \left( \frac{(n+1)^{1-s}}{2(1-s)} \right) (m_{n+1}^v - \bar{m}_{n+1}^v), \\ \Sigma_{n+1}^v &= \left( \frac{n}{n+1} \right)^{1-\delta} \Sigma_n + \frac{1-\delta}{(n+1)^{\delta+s}} \exp \left( -\frac{(n+1)^{1-s}}{1-s} \right) V_{n+1}^v \otimes V_{n+1}^v. \end{aligned}$$

## 5.2 A short simulation study

We consider from now that  $X$  is a random variable taking values in  $\mathbb{R}^d$ , with  $d \geq 3$ , and following a uniform law on the unit sphere  $\mathcal{S}^d$ . Then, the geometric median  $m$  is equal to 0, and assumptions **(H1)** and **(H2)** are verified (see Lemma A.1 in [Godichon-Baggioni and Portier \(2016\)](#)). Moreover, the Hessian of the functional  $G_0$  at  $m$  verifies

$$\Gamma_m = \mathbb{E} \left[ \frac{1}{\|X\|} \left( I_d - \frac{X}{\|X\|} \otimes \frac{X}{\|X\|} \right) \right] = I_d - \mathbb{E} [X \otimes X] = \frac{d-1}{d} I_d.$$

Finally, the asymptotic variance of the averaged estimates verified

$$\Sigma = \Gamma_m^{-1} \mathbb{E} \left[ \frac{X}{\|X\|} \otimes \frac{X}{\|X\|} \right] \Gamma_m^{-1} = \frac{d}{(d-1)^2} I_d.$$

In what follows, we take a dimension  $d = 100$  and a stepsequence  $(\gamma_n)$  verifying for all  $n \geq 1$ ,  $\gamma_n := n^{-2/3}$ . First, let us study the quality of the Gaussian approximation of  $Q_n$ , where  $Q_n := \sqrt{n} \frac{d-1}{\sqrt{d}} (\bar{m}_n - m)$ . Figure 1 seems to confirm Theorem 3.2 since we can see that the estimated density of each coordinate is closed to the density of  $\mathcal{N}(0, 1)$ , which is

also confirmed by a Kolmogorov-Smirnov test.

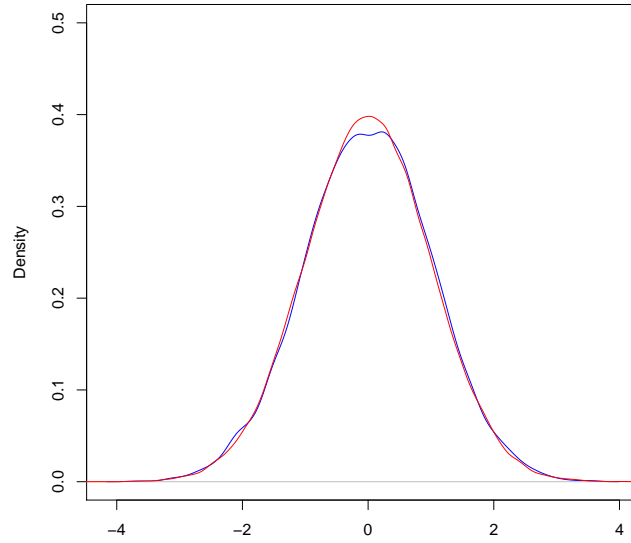


Figure 1: Estimated density of a coordinate of  $Q_{5000}$  (in blue) compared to the standard gaussian density (in red).

In Figures 2 and 3, we consider the evolution of the quadratic mean error, for the Frobenius norm as well as for the spectral norm, of the estimates  $(\Sigma_n)$  of  $\Sigma$  defined by (12), with regard to the sample size. It confirms the fact that the estimates of the asymptotic variance converge quickly, and so, even for moderate dimensions and sample sizes. The results are obtained with the help of 100 generated samples.

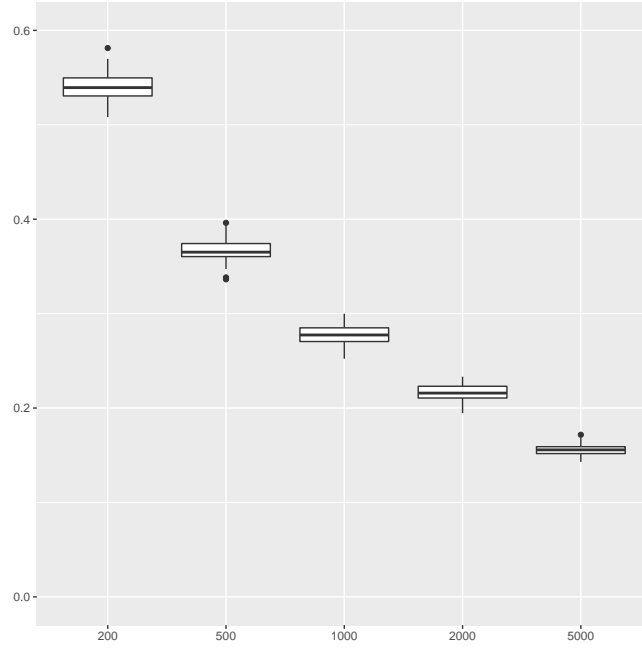


Figure 2: Evolution of the quadratic error of the estimation of the asymptotic variance  $\Sigma$  with respect to the Frobenius norm.

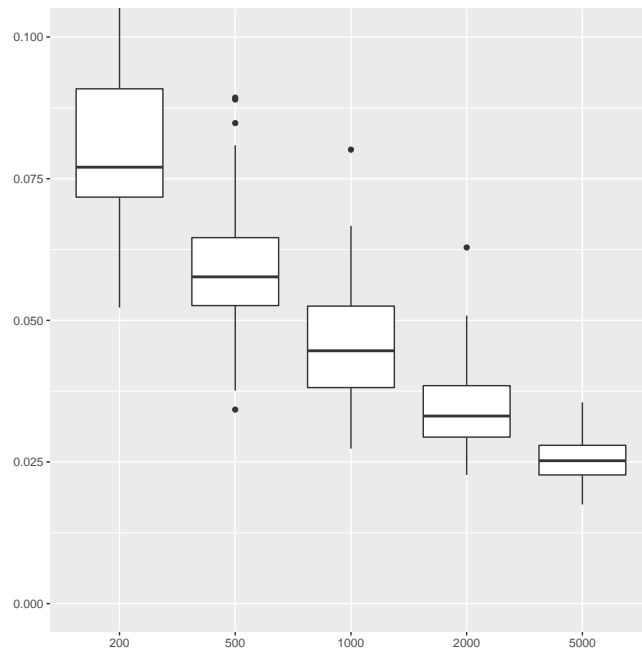


Figure 3: Evolution of the quadratic error of the estimation of the asymptotic variance  $\Sigma$  with respect to the spectral norm.

## 6 Proofs

### 6.1 Some decompositions of the algorithms

In order to simplify the proofs, let us now give some decompositions of the algorithms.

#### 6.1.1 The Robbins-Monro algorithm

Let us recall that the stochastic gradient algorithm can be written as

$$m_{n+1} - m = m_n - m - \gamma_n \Phi(m_n) + \gamma_n \xi_{n+1}.$$

Linearizing the gradient, it comes

$$m_{n+1} - m = (I_H - \gamma_n \Gamma_m)(m_n - m) + \gamma_n \xi_{n+1} - \gamma_n \delta_n, \quad (16)$$

where  $\delta_n := \Gamma_m(m_n - m) - \Phi(m_n)$  is the remainder term in the Taylor's expansion of the gradient. Thanks to previous decomposition and with the help of an induction (see [Duflo \(1996\)](#) or [Duflo \(1997\)](#) for instance), one can check that for all  $n \geq 1$ ,

$$m_n - m = \beta_{n-1}(m_1 - m) - \beta_{n-1} \sum_{k=1}^{n-1} \gamma_k \beta_k^{-1} \delta_k + \beta_{n-1} \sum_{k=1}^{n-1} \gamma_k \beta_k^{-1} \xi_{k+1}, \quad (17)$$

with  $\beta_n := \prod_{k=1}^n (I_H - \gamma_k \Gamma_m)$  for all  $n \geq 1$  and  $\beta_0 := I_H$ . Finally, the asymptotic variance can be seen as the almost sure limit of the sequence of random variables  $(\Gamma_m^{-1} \xi_n \otimes \Gamma_m^{-1} \xi_n)_n$  (see the proof of Theorem 3.2). Then, in order to prove the convergence of the estimates, we need to exhibit this sequence. In this aim, one can rewrite equation (16) as

$$m_n - m = \frac{T_n}{\gamma_n} - \frac{T_{n+1}}{\gamma_n} + \Xi_{n+1} - \Delta_n, \quad (18)$$

with

$$T_n := \Gamma_m^{-1}(m_n - m), \quad \Xi_{n+1} := \Gamma_m^{-1}(\xi_{n+1}), \quad \Delta_n := \Gamma_m^{-1}(\delta_n).$$

#### 6.1.2 The averaged algorithm

Summing equalities (18) and dividing by  $n$ , we obtain the following decomposition of the averaged estimator

$$\bar{m}_n - m = \frac{1}{n} \sum_{k=1}^n \left( \frac{T_k}{\gamma_k} - \frac{T_{k+1}}{\gamma_k} \right) - \frac{1}{n} \sum_{k=1}^n \Delta_k + \frac{1}{n} \sum_{k=1}^n \Xi_{k+1}. \quad (19)$$

Finally, by linearity and applying an Abel's transform to the first term on the right-hand side of previous equality (see [Delyon and Juditsky \(1992\)](#) or [Delyon and Juditsky \(1993\)](#) for

instance),

$$\begin{aligned}\Gamma_m(\bar{m}_n - m) &= \frac{m_1 - m}{n\gamma_1} - \frac{m_{n+1} - m}{n\gamma_n} + \frac{1}{n} \sum_{k=2}^n \left( \frac{1}{\gamma_k} - \frac{1}{\gamma_{k-1}} \right) (m_k - m) - \frac{1}{n} \sum_{k=1}^n \delta_k \\ &\quad + \frac{1}{n} \sum_{k=1}^n \zeta_{k+1}.\end{aligned}\tag{20}$$

### 6.1.3 The recursive estimator of the asymptotic variance

In order to simplify the proof of Theorem 4.1, we will introduce a new estimator of the variance. In this aim, let us now introduce the sequences  $(a_n)_{n \geq 1}$  and  $(b_n)_{n \geq 1}$  defined for all  $n \geq 1$  by  $a_n := \exp\left(\frac{n^{1-s}}{2(1-s)}\right)$  and  $b_n := \sum_{k=1}^n a_k^2$ . Then, thanks to decomposition (18), let

$$\begin{aligned}\bar{T}_n &:= \frac{1}{\sqrt{b_n}} \sum_{k=1}^n a_k (m_k - m) \\ &= \frac{1}{\sqrt{b_n}} \left( \sum_{k=1}^n \frac{a_k}{\gamma_k} (T_k - T_{k+1}) + \sum_{k=1}^n a_k \Delta_k + \sum_{k=1}^n a_k \Xi_{k+1} \right) \\ &=: \frac{1}{\sqrt{b_n}} (A_{1,n} + A_{2,n} + M_{n+1}).\end{aligned}\tag{21}$$

In order to simplify several proofs, we now give  $L^p$  upper bounds of the terms on the right-hand side of previous equality.

**Lemma 6.1.** *Suppose assumptions (A1) to (A5b) hold. Then, for all positive integer  $p$ ,*

$$\begin{aligned}\mathbb{E} \left[ \left\| \sum_{k=1}^n \frac{a_k}{\gamma_k} (T_k - T_{k+1}) \right\|^{2p} \right] &= O \left( \exp \left( \frac{pn^{1-s}}{1-s} \right) n^{p\alpha} \right), \\ \mathbb{E} \left[ \left\| \sum_{k=1}^n a_k \Delta_k \right\|^{2p} \right] &= O \left( \exp \left( \frac{pn^{1-s}}{1-s} \right) n^{p(s-\alpha)} \right), \\ \mathbb{E} \left[ \left\| \sum_{k=1}^n a_k \Xi_{k+1} \right\|^{2p} \right] &= O \left( \exp \left( \frac{pn^{1-s}}{1-s} \right) n^{ps} \right)\end{aligned}$$

The proof of this lemma as well as an analogous lemma which gives the asymptotic almost sure behavior of these terms are given in Appendix. We can now introduce the following estimator

$$\bar{\Sigma}_n = \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} T_k \otimes T_k,\tag{22}$$

and one can decompose  $\Sigma_n$  as follows:

$$\begin{aligned}\Sigma_n - \Sigma &= \Sigma_n - \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \left(\sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m)\right) \otimes \left(\sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m)\right) \\ &\quad + \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \left(\sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m)\right) \otimes \left(\sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m)\right) - \bar{\Sigma}_n \\ &\quad + \bar{\Sigma}_n - \Sigma.\end{aligned}$$

## 6.2 Proof of Theorem 3.1

Let us recall that the Robbins-Monro algorithm can be written for all  $n \geq 1$  as (see (17))

$$m_n - m = \beta_{n-1} (m_1 - m) - \beta_{n-1} \sum_{k=1}^{n-1} \gamma_k \beta_k^{-1} \delta_k + \beta_{n-1} \sum_{k=1}^{n-1} \gamma_k \beta_k^{-1} \zeta_{k+1}.$$

It was proven in [Godichon-Baggioni \(2016b\)](#) that under assumptions **(A1)** to **(A5a)**, for all  $\gamma > 0$ ,

$$\begin{aligned}\frac{1}{\sqrt{\gamma_n}} \left\| \beta_{n-1} (m_1 - m) - \beta_{n-1} \sum_{k=1}^{n-1} \gamma_k \beta_k^{-1} \delta_k \right\| &= O\left(\frac{\|m_n - m\|^2}{\sqrt{\gamma_n}}\right) \quad a.s., \\ &= o\left(\frac{(\ln n)^\gamma}{n^{\alpha/2}}\right) \quad a.s.\end{aligned}$$

Then, we just have to apply Theorem 5.1 in [Jakubowski \(1988\)](#) to the last term on the right-hand side of equality (17). More precisely, let  $(e_i)_{i \in I}$  be an orthonormal basis of  $H$  composed of eigenvectors of  $\Gamma_m$  and let  $\psi'_{i,j} := \langle \Sigma_{RM} e_i, e_j \rangle$  for all  $i, j \in I$ , we have to prove that the following equalities are verified.

$$\forall \eta > 0, \quad \lim_{n \rightarrow \infty} \mathbb{P} \left( \sup_{1 \leq k \leq n} \frac{1}{\sqrt{\gamma_n}} \left\| \beta_n \beta_k^{-1} \gamma_k \zeta_{k+1} \right\| > \eta \right) = 0, \quad (23)$$

$$\lim_{n \rightarrow \infty} \frac{1}{\gamma_n} \sum_{k=1}^n \left\langle \beta_n \beta_k^{-1} \gamma_k \zeta_{k+1}, e_i \right\rangle \left\langle \beta_n \beta_k^{-1} \gamma_k \zeta_{k+1}, e_j \right\rangle = \psi'_{i,j} \quad a.s., \quad \forall i, j \in I, \quad (24)$$

$$\forall \epsilon > 0, \quad \lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \frac{1}{\gamma_n} \sum_{k=1}^n \sum_{j=N}^{\infty} \left\langle \beta_n \beta_k^{-1} \gamma_k \zeta_{k+1}, e_j \right\rangle^2 > \epsilon \right) = 0. \quad (25)$$

**Proof of (23).** Let  $\eta > 0$ , applying Markov's inequality,

$$\begin{aligned}\mathbb{P} \left( \sup_{1 \leq k \leq n} \frac{1}{\sqrt{\gamma_n}} \left\| \beta_n \beta_k^{-1} \gamma_k \zeta_{k+1} \right\| > \eta \right) &\leq \sum_{k=1}^n \mathbb{P} \left( \frac{1}{\sqrt{\gamma_n}} \left\| \beta_n \beta_k^{-1} \gamma_k \zeta_{k+1} \right\| > \eta \right) \\ &\leq \frac{1}{\eta^4 \gamma_n^2} \sum_{k=1}^n \mathbb{E} \left[ \left\| \beta_n \beta_k^{-1} \gamma_k \zeta_{k+1} \right\|^4 \right].\end{aligned}$$

First, since each eigenvalue  $\lambda$  of  $\Gamma_m$  verifies  $0 < \lambda_{\min} \leq \lambda \leq C$ , there is a rank  $n_\alpha$  such that

for all positive integer  $k, n$  verifying  $n_\alpha \leq k \leq n$ ,

$$\left\| \beta_n \beta_k^{-1} \right\|_{op} \leq \prod_{j=k+1}^n \|I_H - \gamma_j \Gamma_m\|_{op} \leq \prod_{j=k+1}^n (1 - \gamma_j \lambda_{\min}) \leq \exp \left( -\lambda_{\min} \sum_{j=k+1}^n \gamma_j \right). \quad (26)$$

For the sake of simplicity, we consider from now that  $n_\alpha = 1$  (one can see the proof of Lemma 3.1 in [Cardot et al. \(2015\)](#) for an analogous and more detailed proof). Then, applying Lemmas 6.6 and 6.8, there is a positive constant  $C$  such that for all  $n \geq 1$ ,

$$\begin{aligned} \sum_{k=1}^{n-1} \mathbb{E} \left[ \left\| \beta_n \beta_k^{-1} \gamma_k \xi_{k+1} \right\|^4 \right] &\leq \sum_{k=1}^n \left\| \beta_n \beta_k^{-1} \right\|_{op}^4 \gamma_k^4 \mathbb{E} \left[ \left\| \xi_{k+1} \right\|^4 \right] \\ &\leq C \sum_{k=1}^n \exp \left( -4\lambda_{\min} \sum_{j=k+1}^n \gamma_j \right) \gamma_k^4 \\ &= O(\gamma_n^3), \end{aligned}$$

which concludes the proof of (23).

**Proof of (24).** Since

$$\sum_{k=1}^n \left\langle \beta_n \beta_k^{-1} \gamma_k \xi_{k+1}, e_i \right\rangle \left\langle \beta_n \beta_k^{-1} \gamma_k \xi_{k+1}, e_j \right\rangle = \sum_{k=1}^n \left\langle \left( \beta_n \beta_k^{-1} \gamma_k \xi_{k+1} \right) \otimes \left( \beta_n \beta_k^{-1} \gamma_k \xi_{k+1} \right) (e_i), e_j \right\rangle,$$

we just have to prove that

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{\gamma_n} \sum_{k=1}^n \left( \beta_n \beta_k^{-1} \gamma_k \xi_{k+1} \right) \otimes \left( \beta_n \beta_k^{-1} \gamma_k \xi_{k+1} \right) - \Sigma_{RM} \right\|_F = 0 \quad a.s. \quad (27)$$

First, note that by linearity

$$\begin{aligned} \sum_{k=1}^n \left( \beta_n \beta_k^{-1} \gamma_k \xi_{k+1} \right) \otimes \left( \beta_n \beta_k^{-1} \gamma_k \xi_{k+1} \right) &= \sum_{k=1}^n \left( \beta_n \beta_k^{-1} \gamma_k \right) (\xi_{k+1} \otimes \xi_{k+1}) \left( \beta_n \beta_k^{-1} \gamma_k \right) \\ &= \sum_{k=1}^n \left( \beta_n \beta_k^{-1} \gamma_k \right) \mathbb{E} [\xi_{k+1} \otimes \xi_{k+1} | \mathcal{F}_k] \left( \beta_n \beta_k^{-1} \gamma_k \right) \\ &\quad + \sum_{k=1}^n \left( \beta_n \beta_k^{-1} \gamma_k \right) \epsilon_{k+1} \left( \beta_n \beta_k^{-1} \gamma_k \right), \end{aligned}$$

with  $\epsilon_{k+1} = \xi_{k+1} \otimes \xi_{k+1} - \mathbb{E} [\xi_{k+1} \otimes \xi_{k+1} | \mathcal{F}_k]$ . Note that  $(\epsilon_k)$  is a sequence of martingale differences adapted to the filtration  $(\mathcal{F}_k)$ . We now prove that the two last terms on the right-hand side of previous equality converge almost surely to 0. First, as in [Godichon-Baggioni \(2016b\)](#) and [Cardot et al. \(2015\)](#), one can check that

$$\lim_{n \rightarrow \infty} \frac{1}{\gamma_n} \left\| \sum_{k=1}^n \left( \beta_n \beta_k^{-1} \gamma_k \right) \epsilon_{k+1} \left( \beta_n \beta_k^{-1} \gamma_k \right) \right\|_F = 0 \quad a.s.$$



Let us now rewrite  $\mathbb{E} [\zeta_{k+1} \otimes \zeta_{k+1} | \mathcal{F}_k]$  as

$$\mathbb{E} [\zeta_{k+1} \otimes \zeta_{k+1} | \mathcal{F}_k] = \Sigma' + (\mathbb{E} [\nabla_{hg}(X_{k+1}, m_k) \otimes \nabla_{hg}(X_{k+1}, m_k) | \mathcal{F}_k] - \Sigma') - \Phi(m_k) \otimes \Phi(m_k).$$

Then, let

$$\begin{aligned} (*) &:= \frac{1}{\gamma_n} \left\| \sum_{k=1}^n (\beta_n \beta_k^{-1} \gamma_k) (\Phi(m_k) \otimes \Phi(m_k)) (\beta_n \beta_k^{-1} \gamma_k) \right\|_F \\ &\leq \frac{1}{\gamma_n} \sum_{k=1}^n \left\| (\beta_n \beta_k^{-1} \gamma_k) (\Phi(m_k) \otimes \Phi(m_k)) (\beta_n \beta_k^{-1} \gamma_k) \right\|_F \\ &\leq \frac{1}{\gamma_n} \sum_{k=1}^n \left\| \beta_n \beta_k^{-1} \gamma_k \Phi(m_k) \right\|^2. \end{aligned}$$

Moreover, since there is a positive constant  $C$  such that for all  $n \geq 1$ ,  $\|\Phi(m_n)\| \leq C\|m_n - m\|$ ,

$$(*) \leq \frac{1}{\gamma_n} \sum_{k=1}^n \left\| \beta_n \beta_k^{-1} \right\|_{op}^2 \gamma_k^2 \|\Phi(m_k)\|^2 \leq \frac{1}{\gamma_n} \sum_{k=1}^n \left\| \beta_n \beta_k^{-1} \right\|_{op}^2 \gamma_k^2 C^2 \|m_k - m\|^2.$$

Thus, applying inequalities (5) and (26) as well as Lemma 6.8, for all  $\beta < \alpha$ ,

$$\frac{1}{\gamma_n} \sum_{k=1}^n \left\| \beta_n \beta_k^{-1} \right\|_{op}^2 \gamma_k^2 C^2 \|m_k - m\|^2 = o \left( \frac{1}{\gamma_n} \sum_{k=1}^n \exp \left( -2\lambda_{\min} \sum_{j=k+1}^n \gamma_j \right) \gamma_k^2 \frac{1}{k^\beta} \right) = o \left( \frac{1}{n^\beta} \right).$$

In the same way,

$$\begin{aligned} &\frac{1}{\gamma_n} \left\| \sum_{k=1}^n (\beta_n \beta_k^{-1} \gamma_k) (\mathbb{E} [\nabla_{hg}(X_{k+1}, m_k) \otimes \nabla_{hg}(X_{k+1}, m_k) | \mathcal{F}_k] - \Sigma') (\beta_n \beta_k^{-1} \gamma_k) \right\|_F \\ &\leq \frac{1}{\gamma_n} \sum_{k=1}^n \left\| \beta_n \beta_k^{-1} \gamma_k \right\|_{op}^2 \left\| \mathbb{E} [\nabla_{hg}(X_{k+1}, m_k) \otimes \nabla_{hg}(X_{k+1}, m_k) | \mathcal{F}_k] - \Sigma' \right\|_F \\ &\leq \frac{1}{\gamma_n} \sum_{k=1}^n \gamma_k^2 \exp \left( -2\lambda_{\min} \sum_{j=k+1}^n \gamma_j \right) \left\| \mathbb{E} [\nabla_{hg}(X_{k+1}, m_k) \otimes \nabla_{hg}(X_{k+1}, m_k) | \mathcal{F}_k] - \Sigma' \right\|_F. \end{aligned}$$

Then, with the help of assumption **(A6a)**, Lemma 6.8 and Toeplitz's lemma, one can check that

$$\lim_{n \rightarrow \infty} \frac{1}{\gamma_n} \left\| \sum_{k=1}^n (\beta_n \beta_k^{-1} \gamma_k) (\mathbb{E} [\nabla_{hg}(X_{k+1}, m_k) \otimes \nabla_{hg}(X_{k+1}, m_k) | \mathcal{F}_k] - \Sigma') (\beta_n \beta_k^{-1} \gamma_k) \right\|_F = 0 \quad a.s.$$

In order to verify equality (27), we have to prove

$$\lim_{n \rightarrow \infty} \frac{1}{\gamma_n} \left\| \sum_{k=1}^n \gamma_k^2 \beta_n \beta_k^{-1} \Sigma' \beta_n \beta_k^{-1} - \Sigma_{RM} \right\| = 0.$$

Let  $(e_i)_{i \in I}$  be an orthonormal basis of  $H$  composed of eigenvectors of  $\Gamma_m$ , and let  $(\lambda_i)_{i \in I}$  be

the set of the associated eigenvalues. Then, let us rewrite  $\nabla_h g(X, m)$  as

$$\nabla_h g(X, m) = \sum_{i \in I} \langle \nabla_h g(X, m), e_i \rangle e_i,$$

and it comes, by linearity and by dominated convergence,

$$\begin{aligned} & \frac{1}{\gamma_n} \sum_{k=1}^n \gamma_k^2 \beta_n \beta_k^{-1} \Sigma' \beta_n \beta_k^{-1} \\ &= \frac{1}{\gamma_n} \sum_{k=1}^n \gamma_k^2 \mathbb{E} \left[ \beta_n \beta_k^{-1} \nabla_h g(X, m) \otimes \beta_n \beta_k^{-1} \nabla_h g(X, m) \right] \\ &= \mathbb{E} \left[ \frac{1}{\gamma_n} \sum_{k=1}^n \gamma_k^2 \left( \sum_{i \in I} \langle \nabla_h g(X, m), e_i \rangle \prod_{j=k+1}^n (1 - \gamma_j \lambda_i) e_i \right) \otimes \left( \sum_{i \in I} \langle \nabla_h g(X, m), e_i \rangle \prod_{j=k+1}^n (1 - \gamma_j \lambda_i) e_i \right) \right]. \end{aligned}$$

In the same way,

$$\begin{aligned} \Sigma_{RM} &= \int_0^\infty e^{-sH} \Sigma' e^{-sH} ds \\ &= \int_0^\infty \mathbb{E} \left[ \left( \sum_{i \in I} \langle \nabla_h g(X, m), e_i \rangle e^{-\lambda_i s} e_i \right) \otimes \left( \sum_{i \in I} \langle \nabla_h g(X, m), e_i \rangle e^{-\lambda_i s} e_i \right) \right] ds \\ &= \mathbb{E} \left[ \int_0^\infty \left( \sum_{i \in I} \langle \nabla_h g(X, m), e_i \rangle e^{-\lambda_i s} e_i \right) \otimes \left( \sum_{i \in I} \langle \nabla_h g(X, m), e_i \rangle e^{-\lambda_i s} e_i \right) ds \right]. \end{aligned}$$

In order to conclude the proof, let us now introduce the following lemma, which allows to give a bound of  $\left\| \frac{1}{\gamma_n} \sum_{k=1}^n \beta_n \beta_k^{-1} \gamma_k \Sigma' \beta_n \beta_k^{-1} \gamma_k - \Sigma_{RM} \right\|_F$ .

**Lemma 6.2.** *There is a positive sequence  $(a_n)$  such that for all  $n \geq 1$  and for all  $i, i' \in I$ ,*

$$-a_n \leq \frac{1}{\gamma_n} \sum_{k=1}^n \gamma_k^2 \prod_{j=k+1}^n (1 - \gamma_j \lambda_i) (1 - \gamma_j \lambda_{i'}) - \int_0^\infty e^{-(\lambda_i + \lambda_{i'})s} ds \leq a_n,$$

and  $\lim_{n \rightarrow \infty} a_n = 0$ .

*Proof.* The proof is given in Appendix. □

Thanks to previous lemma, let

$$\begin{aligned} (*) &= \left\| \frac{1}{\gamma_n} \sum_{k=1}^n \beta_n \beta_k^{-1} \gamma_k \Sigma' \beta_n \beta_k^{-1} \gamma_k - \Sigma_{RM} \right\|_F \\ &\leq \mathbb{E} \left[ \sqrt{\sum_{i, i' \in I} \left( \frac{1}{\gamma_n} \sum_{k=1}^n \gamma_k^2 \prod_{j=k+1}^n (1 - \gamma_j \lambda_i) (1 - \gamma_j \lambda_{i'}) - \frac{1}{\lambda_i + \lambda_{i'}} \right)^2 \langle \nabla_h g(X, m), e_i \rangle^2 \langle \nabla_h g(X, m), e_{i'} \rangle^2} \right] \\ &\leq a_n \mathbb{E} \left[ \sqrt{\sum_{i, i' \in I} \langle \nabla_h g(X, m), e_i \rangle^2 \langle \nabla_h g(X, m), e_{i'} \rangle^2} \right] \\ &= a_n \mathbb{E} \left[ \sum_{i \in I} \langle \nabla_h g(X, m), e_i \rangle^2 \right]. \end{aligned}$$

Under assumption **(A5a)**,

$$\begin{aligned} \left\| \frac{1}{\gamma_n} \sum_{k=1}^n \beta_n \beta_k^{-1} \gamma_k \Sigma' \beta_n \beta_k^{-1} \gamma_k - \Sigma_{RM} \right\|_F &\leq a_n \mathbb{E} \left[ \sum_{i \in I} \langle \nabla_h g(X, m), e_i \rangle^2 \right] \\ &= a_n \mathbb{E} \left[ \|\nabla_h g(X, m)\|^2 \right] \\ &\leq L_1 a_n. \end{aligned}$$

Since  $a_n$  converges to 0, this concludes the proof of inequality (24).

**Proof of inequality (25)** Let  $\epsilon > 0$ , applying Markov's inequality,

$$\begin{aligned} \mathbb{P} \left( \frac{1}{\gamma_n} \sum_{k=1}^n \sum_{j=N}^{\infty} \langle \beta_n \beta_k^{-1} \gamma_k \xi_{k+1}, e_j \rangle > \epsilon \right) &\leq \frac{1}{\gamma_n \epsilon^2} \sum_{k=1}^n \sum_{j=N}^{\infty} \mathbb{E} \left[ \langle \beta_n \beta_k^{-1} \gamma_k \xi_{k+1}, e_j \rangle^2 \right] \\ &= \frac{1}{\gamma_n \epsilon^2} \sum_{k=1}^n \sum_{j=N}^{\infty} \mathbb{E} \left[ \mathbb{E} \left[ \langle \beta_n \beta_k^{-1} \xi_{k+1} \otimes \beta_n \beta_k^{-1} \xi_{k+1} | \mathcal{F}_k \rangle (e_j), e_j \rangle^2 \right] \right] \\ &= \frac{1}{\epsilon^2} \sum_{j=N}^{\infty} \frac{1}{\gamma_n} \sum_{k=1}^n \mathbb{E} \left[ \mathbb{E} \left[ \langle \beta_n \beta_k^{-1} \xi_{k+1} \otimes \beta_n \beta_k^{-1} \xi_{k+1} | \mathcal{F}_k \rangle (e_j), e_j \rangle^2 \right] \right] \end{aligned}$$

Since  $\frac{1}{\gamma_n} \sum_{k=1}^n (\beta_n \beta_k^{-1} \gamma_k \xi_{k+1}) \otimes (\beta_n \beta_k^{-1} \gamma_k \xi_{k+1})$  converges almost surely to  $\Sigma_{RM}$  with respect to the Frobenius norm and by dominated convergence,

$$\limsup_n \mathbb{P} \left( \frac{1}{\gamma_n} \sum_{k=1}^n \sum_{j=N}^{\infty} \langle \beta_n \beta_k^{-1} \gamma_k \xi_{k+1}, e_j \rangle > \epsilon \right) \leq \frac{1}{\epsilon} \sum_{j=N}^{\infty} \langle \Sigma_{RM}(e_j), e_j \rangle.$$

Moreover, since

$$\sum_{j=1}^{\infty} \langle \Sigma_{RM}(e_j), e_j \rangle = \left\| \int_0^{\infty} e^{-sH} \Sigma' e^{-sH} ds \right\|_F \leq \frac{1}{2\lambda_{\min}} \|\Sigma'\|_F \leq \frac{L_1}{2\lambda_{\min}},$$

and since  $\langle \Sigma_{RM}(e_j), e_j \rangle \geq 0$  for all  $j \in I$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{\epsilon} \sum_{j=N}^{\infty} \langle \Sigma_{RM}(e_j), e_j \rangle = 0,$$

which concludes the proof.

### 6.3 Proof of Theorem 3.2

*Proof of Theorem 3.2.* Let us recall that the averaged algorithm can be written as

$$\begin{aligned} \Gamma_m(\bar{m}_n - m) &= \frac{m_1 - m}{n\gamma_1} - \frac{m_{n+1} - m}{n\gamma_n} + \frac{1}{n} \sum_{k=2}^n \left( \frac{1}{\gamma_k} - \frac{1}{\gamma_{k-1}} \right) (m_k - m) - \frac{1}{n} \sum_{k=1}^n \delta_k \\ &\quad + \frac{1}{n} \sum_{k=1}^n \xi_{k+1}. \end{aligned}$$

It is proven in [Godichon-Baggioni \(2016b\)](#) that

$$\begin{aligned}\frac{\|m_1 - m\|}{\sqrt{n}\gamma_1} &= o(1) \quad a.s., \\ \frac{\|m_{n+1} - m\|}{\sqrt{n}\gamma_n} &= o(1) \quad a.s., \\ \frac{1}{\sqrt{n}} \left\| \sum_{k=2}^n \left( \frac{1}{\gamma_k} - \frac{1}{\gamma_{k-1}} \right) (m_k - m) \right\| &= o(1) \quad a.s., \\ \frac{1}{\sqrt{n}} \left\| \sum_{k=1}^n \delta_k \right\| &= o(1) \quad a.s.\end{aligned}$$

In order to get the asymptotic normality of the martingale term  $(\frac{1}{n} \sum_{k=1}^n \tilde{\zeta}_{k+1})$ , let us check that assumptions of Theorem 5.1 in [Jakubowski \(1988\)](#) are fulfilled, i.e. let  $(e_i)_{i \in I}$  be an orthonormal basis of  $H$  and  $\psi_{i,j} := \langle \Sigma' e_i, e_j \rangle$  for all  $i, j \in I$ , we have to verify

$$\forall \eta > 0, \quad \lim_{n \rightarrow \infty} \mathbb{P} \left( \sup_{1 \leq k \leq n} \frac{1}{\sqrt{n}} \|\tilde{\zeta}_{k+1}\| > \eta \right) = 0, \quad (28)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \langle \tilde{\zeta}_{k+1}, e_i \rangle \langle \tilde{\zeta}_{k+1}, e_j \rangle = \psi_{i,j} \quad a.s., \quad \forall i, j \in I, \quad (29)$$

$$\forall \epsilon > 0, \quad \lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \frac{1}{n} \sum_{k=1}^n \sum_{j=N}^{\infty} \langle \tilde{\zeta}_{k+1}, e_j \rangle^2 > \epsilon \right) = 0. \quad (30)$$

**Proof of (28)** Let  $\eta > 0$ , applying Markov's inequality,

$$\begin{aligned}\mathbb{P} \left( \sup_{1 \leq k \leq n} \frac{1}{\sqrt{n}} \|\tilde{\zeta}_{k+1}\| > \eta \right) &\leq \sum_{k=1}^n \mathbb{P} \left( \frac{1}{\sqrt{n}} \|\tilde{\zeta}_{k+1}\| > \eta \right) \\ &\leq \frac{1}{n^2 \eta^4} \sum_{k=1}^n \mathbb{E} [\|\tilde{\zeta}_{k+1}\|^4].\end{aligned}$$

Then, applying Lemma 6.6, there is a positive constant  $C$  such that

$$\mathbb{P} \left( \sup_{1 \leq k \leq n} \frac{1}{\sqrt{n}} \|\tilde{\zeta}_{k+1}\| > \eta \right) \leq \frac{1}{n^2 \eta^4} \sum_{k=1}^n C = \frac{C}{n \eta^4}.$$

**Proof of (29).** First, note that

$$\frac{1}{n} \sum_{k=1}^n \tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} = \frac{1}{n} \sum_{k=1}^n \mathbb{E} [\tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} | \mathcal{F}_k] + \frac{1}{n} \sum_{k=1}^n \epsilon_{k+1},$$

with  $\epsilon_{k+1} := \tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} - \mathbb{E} [\tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} | \mathcal{F}_k]$ . Remark that  $(\epsilon_n)$  is a sequence of martingale differences adapted to the filtration  $(\mathcal{F}_n)$ , and one can check that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \epsilon_{k+1} = 0 \quad a.s.$$

Let us now prove that the sequence of operators  $(\mathbb{E} [\tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} | \mathcal{F}_k])$  converges almost surely to  $\Sigma'$ , with respect to the Frobenius norm. Note that

$$\begin{aligned} \|\mathbb{E} [\tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} | \mathcal{F}_k] - \Sigma'\| &= \|\mathbb{E} [\nabla_{hg}(X_{k+1}, m_k) \otimes \nabla_{hg}(X_{k+1}, m_k) | \mathcal{F}_k] - \Sigma' - \Phi(m_k) \otimes \Phi(m_k)\|_F \\ &\leq \|\mathbb{E} [\nabla_{hg}(X_{k+1}, m_k) \otimes \nabla_{hg}(X_{k+1}, m_k) | \mathcal{F}_k] - \Sigma'\|_F + \|\Phi(m_k) \otimes \Phi(m_k)\|_F. \end{aligned}$$

Then, thanks to assumption **(A6a)**, since  $\|\Phi(m_k)\| \leq C \|m_k - m\|$  and since  $(m_k)$  converges to  $m$  almost surely (see [Godichon-Baggioni \(2016b\)](#)),

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\mathbb{E} [\nabla_{hg}(X_{k+1}, m_k) \otimes \nabla_{hg}(X_{k+1}, m_k) | \mathcal{F}_k] - \Sigma'\|_F &= 0 \quad a.s., \\ \lim_{n \rightarrow \infty} \|\Phi(m_k) \otimes \Phi(m_k)\|_F &= \lim_{n \rightarrow \infty} \|\Phi(m_k)\|^2 = 0 \quad a.s. \end{aligned}$$

In a particular case, for all  $i, j \in I$ ,

$$\lim_{k \rightarrow \infty} \langle \mathbb{E} [\tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} | \mathcal{F}_k] (e_i), e_j \rangle = \psi_{i,j} := \langle \Sigma' (e_i), e_j \rangle \quad a.s.$$

Thus, applying Toeplitz's lemma,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \langle \mathbb{E} [\tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} | \mathcal{F}_k] (e_i), e_j \rangle = \psi_{i,j} \quad a.s.$$

Finally, for all  $i, j \in I$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \langle \tilde{\zeta}_{k+1}, e_i \rangle \langle \tilde{\zeta}_{k+1}, e_j \rangle &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \langle \tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} (e_i), e_j \rangle \\ &= \psi_{i,j} \quad a.s. \end{aligned}$$

**Proof of (30).** Let  $\epsilon > 0$ , applying Markov's inequality,

$$\begin{aligned} \mathbb{P} \left( \frac{1}{n} \sum_{k=1}^n \sum_{j=N}^{\infty} \langle \tilde{\zeta}_{k+1}, e_j \rangle > \epsilon \right) &\leq \frac{1}{n\epsilon^2} \sum_{k=1}^n \sum_{j=N}^{\infty} \mathbb{E} \left[ \langle \tilde{\zeta}_{k+1}, e_j \rangle^2 \right] \\ &= \frac{1}{n\epsilon^2} \sum_{k=1}^n \sum_{j=N}^{\infty} \mathbb{E} \left[ \mathbb{E} \left[ \langle \tilde{\zeta}_{k+1}, e_j \rangle^2 | \mathcal{F}_k \right] \right]. \end{aligned}$$

Since for all  $j \in I$ ,  $\langle \tilde{\zeta}_{k+1}, e_j \rangle^2 = \langle \tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} (e_j), e_j \rangle$ , and by linearity

$$\begin{aligned} \mathbb{P} \left( \frac{1}{n} \sum_{k=1}^n \sum_{j=N}^{\infty} \langle \tilde{\zeta}_{k+1}, e_j \rangle > \epsilon \right) &\leq \frac{1}{\epsilon^2} \sum_{j=N}^{\infty} \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ \mathbb{E} \left[ \langle \tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} (e_j), e_j \rangle | \mathcal{F}_k \right] \right] \\ &= \frac{1}{\epsilon^2} \sum_{j=N}^{\infty} \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ \langle \mathbb{E} [\tilde{\zeta}_{k+1} \otimes \tilde{\zeta}_{k+1} | \mathcal{F}_k] (e_j), e_j \rangle \right]. \end{aligned}$$

Since  $\mathbb{E} [\zeta_{k+1} \otimes \zeta_{k+1} | \mathcal{F}_k]$  converges almost surely to  $\Sigma'$  and by dominated convergence,

$$\limsup_n \mathbb{P} \left( \frac{1}{n} \sum_{k=1}^n \sum_{j=N}^{\infty} \langle \zeta_{k+1}, e_j \rangle > \epsilon \right) \leq \frac{1}{\epsilon} \sum_{j=N}^{\infty} \langle \Sigma'(e_j), e_j \rangle.$$

Moreover, since  $\Sigma' = \mathbb{E} [\nabla_h g(X, m) \otimes \nabla_h g(X, m)]$ , thanks to assumption **(A5a)**,

$$\sum_{j=1}^{\infty} \langle \Sigma'(e_j), e_j \rangle = \|\mathbb{E} [\nabla_h g(X, m) \otimes \nabla_h g(X, m)]\|_F \leq \mathbb{E} [\|\nabla_h g(X, m)\|^2] \leq L_1.$$

Thus, since for all  $j \in I$ ,  $\langle \Sigma'(e_j), e_j \rangle \geq 0$ ,

$$\lim_{N \rightarrow \infty} \sum_{j=N}^{\infty} \langle \Sigma'(e_j), e_j \rangle = 0,$$

which concludes the proof.  $\square$

## 6.4 Proof of Theorem 4.1

Let us recall that equation (12) can be written as

$$\begin{aligned} \Sigma_n - \Sigma &= \Sigma_n - \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \\ &\quad + \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) - \bar{\Sigma}_n \\ &\quad + \bar{\Sigma}_n - \Sigma. \end{aligned} \tag{31}$$

In order to prove Theorem 4.1, we just have to give the rates of convergence of the terms on the right-hand side of previous equality. The following lemma gives the almost sure and the rate of convergence in quadratic mean of the first term on the right-hand side of previous equality.

**Lemma 6.3.** *Suppose assumptions **(A1)** to **(A5a')** and **(A6b)** hold. Then, for all  $\gamma > 0$ ,*

$$\left\| \Sigma_n - \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} e^{-\frac{k^{1-s}}{1-s}} \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \right\|_F^2 = o\left(\frac{(\ln n)^\gamma}{n^{1-s}}\right) \quad a.s.$$

Moreover, suppose assumption **(A5b)** holds too. Then,

$$\mathbb{E} \left[ \left\| \Sigma_n - \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} e^{-\frac{k^{1-s}}{1-s}} \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \right\|_F^2 \right] = O\left(\frac{1}{n^{1-s}}\right).$$

The proof is given in Appendix. The following lemma gives the almost sure and the rate of convergence in quadratic mean of the second term on the right-hand side of equality (31).

**Lemma 6.4.** Suppose assumptions (A1) to (A5a') and (A6b) hold. Then, for all  $\gamma > 0$ ,

$$\left\| \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} e^{-\frac{k^{1-s}}{1-s}} \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) - \bar{\Sigma}_n \right\|_F^2 = o\left(\frac{(\ln n)^\gamma}{n^{2(1-s)}}\right) \quad a.s.$$

Moreover, suppose assumption (A5b) holds too. Then

$$\mathbb{E} \left[ \left\| \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} e^{-\frac{k^{1-s}}{1-s}} \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) - \bar{\Sigma}_n \right\|_F^2 \right] = O\left(\frac{1}{n^{2(1-s)}}\right).$$

The proof is given in Appendix. Finally, the following Proposition gives the almost sure and the rate of convergence in quadratic mean of the last term on the right-hand side of equality (31).

**Proposition 6.1.** Suppose assumptions (A1) to (A5a') and (A6b) hold. Then, there is a positive constant  $\gamma$  such that

$$\|\bar{\Sigma}_n - \Sigma\|_F^2 = o\left(\frac{(\ln n)^\delta}{n^{1-s}}\right) \quad a.s.$$

Suppose assumption (A5b) holds too. Then, there is a positive constant  $C$  such that for all  $n \geq 1$ ,

$$\mathbb{E} \left[ \|\bar{\Sigma}_n - \Sigma\|_F^2 \right] \leq \frac{C}{n^{1-s}}.$$

*Proof of Proposition 6.1.* Applying equality (2), one can check that

$$\begin{aligned} \|\bar{\Sigma}_n - \Sigma\|_F &\leq \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \|A_{1,k}\|^2 + \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \|A_{2,k}\|^2 \\ &\quad + 2 \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \|A_{1,k}\| \|A_{2,k}\| + 2 \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \|A_{1,k}\| \|M_{k+1}\| \\ &\quad + 2 \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \|A_{2,k}\| \|M_{k+1}\| + \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \left( \frac{1}{b_k} M_{k+1} \otimes M_{k+1} - \Sigma \right) \right\|_F, \end{aligned} \quad (32)$$

where  $A_{1,k}, A_{2,k}, M_{k+1}$  are defined in (21). The following Lemma gives the rate of convergence in quadratic mean of the first terms on the right-hand side of previous inequality.

**Lemma 6.5.** Suppose Assumptions (A1) to (A6b) hold. Then, for all  $i, j \in \{1, 2\}$ ,

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \|A_{i,k}\| \|A_{j,k}\| \right)^2 \right] &= o\left(\frac{1}{n^{1-s}}\right), \\ \mathbb{E} \left[ \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \|A_{i,k}\| \|M_{k+1}\| \right)^2 \right] &= o\left(\frac{1}{n^{1-s}}\right). \end{aligned}$$

The proof of this lemma as well as its "almost sure version" are given in Appendix.

Then, we just have to bound the last term on the right-hand side of inequality (32). First let us decompose  $M_{k+1} \otimes M_{k+1}$  as

$$\begin{aligned} M_{k+1} \otimes M_{k+1} &= \sum_{j=1}^k a_j^2 \Xi_{j+1} \otimes \Xi_{j+1} + \sum_{j=1}^k a_j \Xi_{j+1} \otimes M_j + \sum_{j=1}^k a_j \Xi_{j+1} \otimes (M_{k+1} - M_{j+1}) \\ &\quad + \sum_{j=1}^k a_j M_j \otimes \Xi_{j+1} + \sum_{j=1}^k a_j (M_{k+1} - M_{j+1}) \otimes \Xi_{j+1}. \end{aligned}$$

Note that for all  $j$ ,  $M_j$  is  $\mathcal{F}_j$ -measurable and  $\mathbb{E} [\Xi_{j+1} \otimes M_j | \mathcal{F}_j] = 0$ . Moreover,

$$\begin{aligned} \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \left( \frac{1}{b_k} M_{k+1} \otimes M_{k+1} - \Sigma \right) &= \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j (\Xi_{j+1} \otimes \Xi_{j+1} - \Sigma) \\ &\quad + \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j \Xi_{j+1} \otimes M_j + \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j \Xi_{j+1} \otimes (M_{k+1} - M_{j+1}) \\ &\quad + \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j M_j \otimes \Xi_{j+1} + \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j (M_{k+1} - M_{j+1}) \otimes \Xi_{j+1}. \end{aligned}$$

The end of the proof consists in giving a bound of the quadratic mean of each term on the right-hand side of previous equality. Note that the almost sure rates of convergence are not proven since it is quite analogous.

**Bounding**  $\mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j \Xi_{j+1} \otimes M_j \right\|_F^2 \right]$ . First, note that

$$\frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j \Xi_{j+1} \otimes M_j = \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \left( \sum_{j=k}^n \frac{1}{k^\delta} \frac{1}{b_k} \right) a_k \Xi_{k+1} \otimes M_k.$$

Moreover, with the help of an integral test for convergence, one can check that there is a positive constant  $C$  such that for all positive integers  $k \leq n$ ,

$$\sum_{j=k}^n \frac{1}{k^\delta} \frac{1}{b_k} \leq \frac{C}{k^\delta} \exp \left( -\frac{k^{1-s}}{(1-s)} \right). \quad (33)$$

Furthermore, since  $(\Xi_{j+1} \otimes M_j)_j$  is a sequence of martingale differences adapted to the filtration  $(\mathcal{F}_j)$ , let

$$\begin{aligned} (*) &:= \mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j \Xi_{j+1} \otimes M_j \right\|_F^2 \right] \\ &= \mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \left( \sum_{j=k}^n \frac{1}{k^\delta} \frac{1}{b_k} \right) a_k \Xi_{k+1} \otimes M_k \right\|_F^2 \right] \\ &= \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \left( \sum_{j=k}^n \frac{1}{k^\delta} \frac{1}{b_k} \right)^2 a_k^2 \mathbb{E} [\|\Xi_{k+1} \otimes M_k\|_F^2] \end{aligned}$$



Then, applying equality (2) and Cauchy-Schwarz's inequality,

$$\begin{aligned}
(*) &\leq \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \left( \sum_{j=k}^n \frac{1}{k^\delta} \frac{1}{b_k} \right)^2 a_k^2 \mathbb{E} \left[ \|\Xi_{k+1}\|^2 \|M_k\|^2 \right] \\
&\leq \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \left( \sum_{j=k}^n \frac{1}{k^\delta} \frac{1}{b_k} \right)^2 a_k^2 \sqrt{\mathbb{E} \left[ \|\Xi_{k+1}\|^4 \right] \mathbb{E} \left[ \|M_k\|^4 \right]}.
\end{aligned}$$

Finally, applying Lemmas 6.1 and 6.6 as well as inequality (33),

$$(*) = O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \frac{1}{k^{2\delta-s}} \right) = O \left( \frac{1}{n^{1-s}} \right).$$

With analogous calculus, one can check

$$\mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j M_j \otimes \Xi_{j+1} \right\|_F^2 \right] = O \left( \frac{1}{n^{1-s}} \right).$$

**Bounding**  $\mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j \Xi_{j+1} \otimes (M_{k+1} - M_j) \right\|_F^2 \right]$ . First, note that

$$\begin{aligned}
\sum_{j=1}^k a_j \Xi_{j+1} \otimes (M_{k+1} - M_j) &= \sum_{j=1}^k \sum_{j'=j+1}^k a_j a_{j'} \Xi_{j+1} \otimes \Xi_{j'+1} \\
&= \sum_{j'=2}^k \sum_{j=1}^{j'-1} a_j a_{j'} \Xi_{j+1} \otimes \Xi_{j'+1}.
\end{aligned}$$

Note that  $\left( \sum_{j=1}^{j'-1} a_j a_{j'} \Xi_{j+1} \otimes \Xi_{j'+1} \right)_{j'}$  is a sequence of martingale differences adapted to the filtration  $(\mathcal{F}_{j'})$ . Furthermore,

$$\begin{aligned}
&\mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j \Xi_{j+1} \otimes (M_{k+1} - M_j) \right\|_F^2 \right] \\
&= \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \frac{1}{k^{2\delta}} \frac{1}{b_k^2} \mathbb{E} \left[ \left\| \sum_{j'=2}^k \sum_{j=1}^{j'-1} a_j a_{j'} \Xi_{j+1} \otimes \Xi_{j'+1} \right\|_F^2 \right] \\
&+ \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \mathbb{E} \left[ \sum_{k=2}^n \sum_{j=1}^{k-1} b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \left\langle \sum_{j''=2}^j \sum_{j'=1}^{j''-1} a_{j'} a_{j''} \Xi_{j'+1} \otimes \Xi_{j''+1}, \sum_{i''=2}^k \sum_{i'=1}^{i''-1} a_{i'} a_{i''} \Xi_{i'+1} \otimes \Xi_{i''+1} \right\rangle \right].
\end{aligned}$$

Then end of the proof consists in bounding the two terms on the right-hand side of previous equality. First, since  $\left( \sum_{j=1}^{j'-1} a_j a_{j'} \Xi_{j+1} \otimes \Xi_{j'+1} \right)_{j'}$  is a sequence of martingale differences

adapted to the filtration  $(\mathcal{F}_{j'})$ , let

$$\begin{aligned} (\star) &:= \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \frac{1}{k^{2\delta}} \frac{1}{b_k^2} \mathbb{E} \left[ \left\| \sum_{j'=2}^k \sum_{j=1}^{j'-1} a_j a_{j'} \Xi_{j+1} \otimes \Xi_{j'+1} \right\|_F^2 \right] \\ &= \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \frac{1}{k^{2\delta}} \frac{1}{b_k^2} \sum_{j'=2}^k \mathbb{E} \left[ \left\| \sum_{j=1}^{j'-1} a_j a_{j'} \Xi_{j+1} \otimes \Xi_{j'+1} \right\|_F^2 \right]. \end{aligned}$$

Then, applying equality (2) and Cauchy-Schwarz's inequality,

$$\begin{aligned} (\star) &= \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \frac{1}{k^{2\delta}} \frac{1}{b_k^2} \sum_{j'=2}^k a_{j'}^2 \mathbb{E} \left[ \left\| \sum_{j=1}^{j'-1} a_j \Xi_{j+1} \right\|^2 \|\Xi_{j'+1}\|^2 \right] \\ &\leq \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \frac{1}{k^{2\delta}} \frac{1}{b_k^2} \sum_{j'=2}^k a_{j'}^2 \sqrt{\mathbb{E} [\|\Xi_{j'+1}\|^4]} \sqrt{\mathbb{E} \left[ \left\| \sum_{j=1}^{j'-1} a_j \Xi_{j+1} \right\|^4 \right]} \end{aligned}$$

Finally, applying Lemma 6.6, 6.7 and 6.1,

$$(\star) = O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \frac{1}{k^{2\delta}} \frac{1}{b_k^2} \sum_{j'=2}^k a_{j'}^4 j'^s \right) = O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \frac{1}{k^{2\delta}} \frac{1}{b_k^2} a_k^4 k^{2s} \right) = O \left( \frac{1}{n^{\min\{2-2\delta, 1\}}} \right).$$

Then, since  $\delta < (1+s)/2$ ,

$$\left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \frac{1}{k^{2\delta}} \frac{1}{b_k^2} \mathbb{E} \left[ \left\| \sum_{j'=2}^k \sum_{j=1}^{j'-1} a_j a_{j'} \Xi_{j+1} \otimes \Xi_{j'+1} \right\|_F^2 \right] = o \left( \frac{1}{n^{1-s}} \right).$$

In the same way, by linearity, let

$$\begin{aligned} (\star\star) &:= \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \mathbb{E} \left[ \sum_{k=2}^n \sum_{j=1}^{k-1} b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \left\langle \sum_{j''=2}^j \sum_{j'=1}^{j''-1} a_{j'} a_{j''} \Xi_{j'+1} \otimes \Xi_{j''+1}, \sum_{i''=2}^k \sum_{i'=1}^{i''-1} a_{i'} a_{i''} \Xi_{i'+1} \otimes \Xi_{i''+1} \right\rangle \right] \\ &= \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=2}^n \sum_{j=1}^{k-1} b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \mathbb{E} \left[ \left\langle \sum_{j''=2}^j \sum_{j'=1}^{j''-1} a_{j'} a_{j''} \Xi_{j'+1} \otimes \Xi_{j''+1}, \sum_{i''=2}^j \sum_{i'=1}^{i''-1} a_{i'} a_{i''} \Xi_{i'+1} \otimes \Xi_{i''+1} \right\rangle_F \right] \\ &+ \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=2}^n \sum_{j=1}^{k-1} b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \mathbb{E} \left[ \left\langle \sum_{j''=2}^j \sum_{j'=1}^{j''-1} a_{j'} a_{j''} \Xi_{j'+1} \otimes \Xi_{j''+1}, \sum_{i''=j+1}^k \sum_{i'=1}^{i''-1} a_{i'} a_{i''} \Xi_{i'+1} \otimes \Xi_{i''+1} \right\rangle_F \right]. \end{aligned}$$

Since  $(\Xi_{i''})$  is a sequence of martingale differences adapted to the filtration  $(\mathcal{F}_{i''})$ ,

$$\begin{aligned}
& \sum_{k=2}^n \sum_{j=1}^{k-1} b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \mathbb{E} \left[ \left\langle \sum_{j''=2}^j \sum_{j'=1}^{j''-1} a_{j'} a_{j''} \Xi_{j'+1} \otimes \Xi_{j''+1}, \sum_{i''=j+1}^k \sum_{i'=1}^{i''-1} a_{i'} a_{i''} \Xi_{i'+1} \otimes \Xi_{i''+1} \right\rangle_F \right] \\
&= \sum_{k=2}^n \sum_{j=1}^{k-1} b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \sum_{j''=2}^j \sum_{j'=1}^{j''-1} \sum_{i''=j+1}^k \sum_{i'=1}^{i''-1} a_{i'} a_{i''} a_{j'} a_{j''} \mathbb{E} \left[ \langle \Xi_{j'+1} \otimes \Xi_{j''+1}, \Xi_{i'+1} \otimes \Xi_{i''+1} \rangle_F \right] \\
&= \sum_{k=2}^n \sum_{j=1}^{k-1} b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \sum_{j''=2}^j \sum_{j'=1}^{j''-1} \sum_{i''=j+1}^k \sum_{i'=1}^{i''-1} a_{i'} a_{i''} a_{j'} a_{j''} \mathbb{E} \left[ \langle \Xi_{j'+1} \otimes \Xi_{j''+1}, \Xi_{i'+1} \otimes \mathbb{E} [\Xi_{i''+1} | \mathcal{F}_{i''}] \rangle_F \right] \\
&= 0.
\end{aligned}$$

Furthermore, since  $(\sum_{j''=2}^j \sum_{j'=1}^{j''-1} a_{j'} a_{j''} \Xi_{j'+1} \otimes \Xi_{j''+1})_{j''}$  is a sequence of martingale differences adapted to the filtration  $(\mathcal{F}_{j''})$  and applying equality (2),

$$\begin{aligned}
(\star\star) &= \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=2}^n \sum_{j=1}^k b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \mathbb{E} \left[ \left\| \sum_{j''=1}^j \sum_{j'=1}^{j''-1} a_{j'} a_{j''} \Xi_{j'+1} \otimes \Xi_{j''+1} \right\|_F^2 \right] \\
&= \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=2}^n \sum_{j=1}^k b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \sum_{j''=1}^j \mathbb{E} \left[ \left\| \sum_{j'=1}^{j''-1} a_{j'} a_{j''} \Xi_{j'+1} \otimes \Xi_{j''+1} \right\|_F^2 \right] \\
&= \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=2}^n \sum_{j=1}^k b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \sum_{j''=1}^j a_{j''}^2 \mathbb{E} \left[ \left\| \sum_{j'=1}^{j''-1} a_{j'} \Xi_{j'+1} \right\|^2 \|\Xi_{j''+1}\|^2 \right].
\end{aligned}$$

Applying Cauchy-Schwarz's inequality as well as Lemmas 6.6 and 6.1,

$$\begin{aligned}
(\star\star) &\leq \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \sum_{j=1}^k b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \sum_{j''=1}^j a_{j''}^2 \sqrt{\mathbb{E} \left[ \left\| \sum_{j'=1}^{j''-1} a_{j'} \Xi_{j'+1} \right\|_F^4 \right] \mathbb{E} [\|\Xi_{j''+1}\|_F^4]} \\
&= O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \sum_{j=1}^k b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} \sum_{j''=1}^j a_{j''}^4 j''^s \right).
\end{aligned}$$

Finally, applying Lemma 6.7,

$$\begin{aligned}
(\star\star) &= O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n \sum_{j=1}^k b_k^{-1} k^{-\delta} b_j^{-1} j^{-\delta} a_j^4 j^{2s} \right) \\
&= O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \sum_{k=1}^n b_k^{-1} k^{-2\delta} k^{2s} a_k^2 \right) \\
&= O \left( \frac{1}{n^{1-s}} \right).
\end{aligned}$$

Thus,

$$\mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j \Xi_{j+1} \otimes (M_{k+1} - M_{j+1}) \right\|_F^2 \right] = O \left( \frac{1}{n^{1-s}} \right).$$

Moreover, with analogous calculus, one can check

$$\mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta} \frac{1}{b_k} \sum_{j=1}^k a_j (M_{k+1} - M_{j+1}) \otimes \Xi_{j+1} \right\|_F^2 \right] = O \left( \frac{1}{n^{1-s}} \right).$$

**Bounding**  $\frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \sum_{j=1}^k a_k^2 (\Xi_{k+1} \otimes \Xi_{k+1} - \Sigma)$ . First, note that

$$\begin{aligned} \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \sum_{j=1}^k a_k^2 (\Xi_{k+1} \otimes \Xi_{k+1} - \Sigma) &= \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \sum_{j=1}^k a_k^2 (\mathbb{E} [\Xi_{k+1} \otimes \Xi_{k+1} | \mathcal{F}_k] - \Sigma) \\ &\quad + \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \sum_{j=1}^k a_k^2 (\Xi_{k+1} \otimes \Xi_{k+1} - \mathbb{E} [\Xi_{k+1} \otimes \Xi_{k+1} | \mathcal{F}_k]) \end{aligned}$$

The end of the proof consists in bounding the quadratic mean of the terms on the right-hand side of previous equality. First, applying Lemma 6.9, let

$$\begin{aligned} (\star) &:= \mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \sum_{j=1}^k a_j^2 (\mathbb{E} [\Xi_{k+1} \otimes \Xi_{k+1} | \mathcal{F}_k] - \Sigma) \right\|_F^2 \right] \\ &\leq \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta b_k} \sqrt{\mathbb{E} \left[ \left\| \sum_{j=1}^k a_j^2 (\mathbb{E} [\Xi_{k+1} \otimes \Xi_{k+1} | \mathcal{F}_k] - \Sigma) \right\|_F^2 \right]} \right)^2 \\ &\leq \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta b_k} \sum_{j=1}^k a_j^2 \sqrt{\mathbb{E} [\|\mathbb{E} [\Xi_{k+1} \otimes \Xi_{k+1} | \mathcal{F}_k] - \Sigma\|_F^2]} \right)^2 \end{aligned}$$

Then, applying inequality (6) and Corollary 6.1,

$$\begin{aligned} (\star) &= O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta b_k} \sum_{j=1}^k a_j^2 \sqrt{\mathbb{E} [\|m_n - m\|^2]} \right)^2 \right) \\ &= O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta b_k} \sum_{j=1}^k a_j^2 j^{-\alpha/2} \right)^2 \right). \end{aligned}$$

Furthermore, thanks to Lemma 6.7,

$$(\star) = O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta b_k} a_k^2 k^{s-\alpha/2} \right)^2 \right) = O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 n^{2-2\delta-\alpha} \right) = O \left( \frac{1}{n^\alpha} \right).$$

Thus, since  $\alpha > 1/2$ ,

$$\mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \sum_{j=1}^k a_j^2 (\mathbb{E} [\Xi_{k+1} \otimes \Xi_{k+1} | \mathcal{F}_k] - \Sigma) \right\|_F^2 \right] = o \left( \frac{1}{n^{1-s}} \right).$$

Moreover, applying Lemma 6.9, let

$$\begin{aligned}
(\star\star) &:= \mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \sum_{j=1}^k a_j^2 (\Xi_{k+1} \otimes \Xi_{k+1} - \mathbb{E} [\Xi_{k+1} \otimes \Xi_{k+1} | \mathcal{F}_k]) \right\|_F^2 \right] \\
&\leq \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta b_k} \sqrt{\mathbb{E} \left[ \left\| \sum_{j=1}^k a_j^2 (\Xi_{k+1} \otimes \Xi_{k+1} - \mathbb{E} [\Xi_{k+1} \otimes \Xi_{k+1} | \mathcal{F}_k]) \right\|_F^2 \right]} \right)^2.
\end{aligned}$$

Furthermore, since  $(\mathbb{E} [\Xi_{k+1} \otimes \Xi_{k+1} | \mathcal{F}_k] - \Xi_{k+1} \otimes \Xi_{k+1})$  is a sequence of martingale differences adapted to the filtration  $(\mathcal{F}_k)$  and applying Lemma 6.6,

$$\begin{aligned}
(\star\star) &\leq \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta b_k} \sqrt{\sum_{j=1}^k a_j^4 \mathbb{E} [\|(\Xi_{k+1} \otimes \Xi_{k+1} - \mathbb{E} [\Xi_{k+1} \otimes \Xi_{k+1} | \mathcal{F}_k])\|_F^2]} \right)^2 \\
&= O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta b_k} \sqrt{\sum_{j=1}^k a_j^4} \right)^2 \right).
\end{aligned}$$

Then, applying Lemma 6.7,

$$(\star\star) = O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta b_k} a_k^2 k^{s/2} \right)^2 \right) = O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n k^{-\delta-s/2} \right)^2 \right) = O \left( \frac{1}{n^{2-s}} \right).$$

Finally,

$$\mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \sum_{j=1}^k a_j^2 (\Xi_{k+1} \otimes \Xi_{k+1} - \mathbb{E} [\Xi_{k+1} \otimes \Xi_{k+1} | \mathcal{F}_k]) \right\|_F^2 \right] = o \left( \frac{1}{n^{1-s}} \right),$$

which concludes the proof.  $\square$

## 6.5 Technical lemmas

In order to simplify the proof, we recall or give some technical lemmas. The following one ensures that the sequence  $(\xi_n)$  admits uniformly bounded  $2p$ -moments.

**Lemma 6.6** (Godichon-Baggioni (2016b)). *Suppose assumptions (A1) to (A5a') hold, there is a positive constant  $K$  such that for all  $n \geq 1$ ,*

$$\mathbb{E} [\|\xi_{n+1}\|^4] \leq K$$

Moreover, suppose assumption (A5b) holds too. Then, for all positive integer  $p$ , there is a positive constant  $K_p$  such that for all  $n \geq 1$ ,

$$\mathbb{E} [\|\xi_{n+1}\|^{2p}] \leq K_p$$

As a particular case, since for all eigenvalue  $\lambda$  of  $\Gamma_m$ ,  $0 < \lambda_{\min} \leq \lambda \leq C$ , for all  $n \geq 1$ ,

$$\mathbb{E} \left[ \|\Xi_{n+1}\|^{2p} \right] \leq K_p \lambda_{\min}^{-2p}.$$

**Corollary 6.1.** Suppose assumptions **(A1)** to **(A6b)** hold. Then, there is a positive constant  $C$  such that for all  $n \geq 1$ ,

$$\|\mathbb{E} [\Xi_{n+1} \otimes \Xi_{n+1} | \mathcal{F}_n] - \Sigma\|_F^2 \leq C \|m_n - m\|^2$$

The proof is not given since it is a direct application of assumption **(A6b)** and Lemma 6.6. The following lemma gives upper bounds of the sums of exponential terms which appears in several proofs.

**Lemma 6.7.** For all constants  $a, b, c$  such that  $a \in (0, 1)$ , there is a positive constant  $C_{a,b,c}$  such that

$$\sum_{k=1}^n k^{-a} k^b \exp \left( ck^{1-a} \right) \leq C_{a,b,c} n^b \exp \left( cn^{1-a} \right).$$

The proof is not given it is a direct application of an integral test for convergence. As a corollary, one can obtain the following bound (lower and upper) of  $b_n$ .

**Corollary 6.2.** There are positive constants  $c, C$  such that for all  $n \geq 1$ ,

$$cn^s \exp \left( \frac{n^{1-s}}{1-s} \right) \leq b_n \leq Cn^s \exp \left( \frac{n^{1-s}}{1-s} \right).$$

The following lemma is really useful in the proof of Theorem 3.1.

**Lemma 6.8** (Cardot and Godichon-Baggioni (2015)). Let  $\alpha, \beta$  be non-negative constants such that  $0 < \alpha < 1$ , and  $(u_n), (v_n)$  be two sequences defined for all  $n \geq 1$  by

$$u_n := \frac{c_u}{n^\alpha}, \quad v_n := \frac{c_v}{n^\beta},$$

with  $c_u, c_v > 0$ . Thus, there is a positive constant  $c_0$  such that for all  $n \geq 1$ ,

$$\sum_{k=1}^n e^{-\sum_{j=k}^n u_j} u_k v_k = O(v_n). \quad (34)$$

Finally, we recall the following results, which enables us to upper bound the  $L^p$  moments of a sum of random variables in normed vector spaces.

**Lemma 6.9** (Godichon-Baggioni (2016a)). Let  $Y_1, \dots, Y_n$  be random variables taking values in a normed vector space such that for all positive constant  $q$  and for all  $k \geq 1$ ,  $\mathbb{E} [\|Y_k\|^q] < \infty$ . Thus, for all constants  $a_1, \dots, a_n$  and for all integer  $p$ ,

$$\mathbb{E} \left[ \left\| \sum_{k=1}^n a_k Y_k \right\|^p \right] \leq \left( \sum_{k=1}^n |a_k| \left( \mathbb{E} [\|Y_k\|^p] \right)^{\frac{1}{p}} \right)^p. \quad (35)$$

## A Appendix

### A.1 Proof of Lemma 6.1

*Proof.* **Bounding**  $\mathbb{E} \left[ \left\| \sum_{k=1}^n \frac{a_k}{\gamma_k} (T_k - T_{k+1}) \right\|^{2p} \right]$ . Applying an Abel's transform,

$$A_{1,n} = \frac{a_1}{\gamma_1} T_1 - \frac{a_n}{\gamma_n} T_{n+1} + \sum_{k=2}^n \left( \frac{a_k}{\gamma_k} - \frac{a_{k-1}}{\gamma_{k-1}} \right) T_k.$$

First,  $\mathbb{E} \left[ \left\| \frac{a_1}{\gamma_1} T_1 \right\|^{2p} \right] = O(1)$ . Moreover, applying inequality (6),

$$\begin{aligned} \mathbb{E} \left[ \left\| \frac{a_n}{\gamma_n} T_{n+1} \right\|^{2p} \right] &\leq \exp \left( \frac{pn^{1-s}}{1-s} \right) c_\gamma^{-1} n^{2p\alpha} \frac{C_p \lambda_{\min}^{-2p}}{n^{p\alpha}} \\ &\leq \exp \left( \frac{pn^{1-s}}{1-s} \right) C_p c_\gamma^{-1} \lambda_{\min}^{-2p} n^{p\alpha}. \end{aligned}$$

Furthermore, one can check that there is a positive constant  $C$  such that for all  $n \geq 1$ ,

$$\left| \frac{a_n}{\gamma_n} - \frac{a_{n-1}}{\gamma_{n-1}} \right| \leq C n^{-s+\alpha} \exp \left( \frac{n^{1-s}}{2(1-s)} \right),$$

and applying Lemma 6.9 and inequality (6),

$$\begin{aligned} \mathbb{E} \left[ \left\| \sum_{k=2}^n \left( \frac{a_k}{\gamma_k} - \frac{a_{k-1}}{\gamma_{k-1}} \right) T_k \right\|^{2p} \right] &\leq \left( \sum_{k=2}^n \left| \frac{a_k}{\gamma_k} - \frac{a_{k-1}}{\gamma_{k-1}} \right| \left( \mathbb{E} [\|T_k\|^{2p}] \right)^{\frac{1}{2p}} \right)^{2p} \\ &\leq C^{2p} C_p \lambda_{\min}^{-2p} \left( \sum_{k=2}^n k^{-s+\alpha} \exp \left( \frac{k^{1-s}}{2(1-s)} \right) k^{-\alpha/2} \right)^{2p}. \end{aligned}$$

Finally, applying Lemma 6.7,

$$\mathbb{E} \left[ \left\| \sum_{k=2}^n \left( \frac{a_k}{\gamma_k} - \frac{a_{k-1}}{\gamma_{k-1}} \right) T_k \right\|^{2p} \right] = O \left( \exp \left( \frac{pn^{1-s}}{1-s} \right) n^{p\alpha} \right).$$

**Bounding**  $\mathbb{E} [\|\sum_{k=1}^n a_k \Delta_k\|^{2p}]$ . Since there is a positive constant  $C_m$  (see [Godichon-Baggioni \(2016b\)](#)) such that for all  $n \geq 1$ ,  $\|\Delta_n\| \leq C_m \|T_n\|^2$ , applying Lemma 6.9 and inequality (6),

$$\begin{aligned} \mathbb{E} \left[ \left\| \sum_{k=1}^n a_k \Delta_k \right\|^{2p} \right] &\leq C_m^{2p} \lambda_{\min}^{-4p} \left( \sum_{k=1}^n a_k \left( \mathbb{E} [\|m_n - m\|^{4p}] \right)^{\frac{1}{2p}} \right)^{2p} \\ &\leq C_m^{2p} \lambda_{\min}^{-4p} C_{2p} \left( \sum_{k=1}^n a_k k^{-\alpha} \right)^{2p} \end{aligned}$$

Applying Lemma 6.7,

$$\mathbb{E} \left[ \left\| \sum_{k=1}^n a_k \Delta_k \right\|^{2p} \right] = O \left( \exp \left( \frac{pn^{1-s}}{1-s} \right) n^{2p(s-\alpha)} \right)$$

**Bounding**  $\mathbb{E} \left[ \left\| \sum_{k=1}^n a_k \Xi_{k+1} \right\|^{2p} \right]$ . First, since  $(\Xi_n)$  is a sequence of martingale differences, and thanks to Lemma 6.7,

$$\begin{aligned} \mathbb{E} \left[ \left\| \sum_{k=1}^n a_k \Xi_{k+1} \right\|^2 \right] &= \sum_{k=1}^n a_k^2 \mathbb{E} \left[ \|\Xi_{k+1}\|^2 \right] \\ &= O \left( \exp \left( \frac{n^{1-s}}{1-s} \right) n^s \right). \end{aligned}$$

With the help of an induction on  $p$  (see the proof of Theorem 4.2 in Godichon-Baggioni (2016a) for instance), one can check that for all integer  $p \geq 1$ ,

$$\mathbb{E} \left[ \left\| \sum_{k=1}^n a_k \Xi_{k+1} \right\|^{2p} \right] = O \left( \exp \left( p \frac{n^{1-s}}{1-s} \right) n^{ps} \right),$$

which concludes the proof.  $\square$

## A.2 Proof of Lemma 6.2

Let  $(\lambda_i)_{i \in I}$  be the eigenvalues of the Hessian  $\Gamma_m$ . First, let

$$c_{n,k} := \prod_{j=k+1}^n (1 - \gamma_j \lambda_i) \prod_{j=k+1}^n (1 - \gamma_j \lambda_{i'}) = \exp \left( \sum_{j=k+1}^n (\ln(1 - \gamma_j \lambda_i) + \ln(1 - \gamma_j \lambda_{i'})) \right)$$

Let us recall that there is a positive constant  $C$  such that for all  $i \in I$ ,  $\lambda_i \leq C$ . Then, let  $n_\alpha$  be an integer such that for all  $k \geq n_\alpha$ ,  $C\gamma_k < 1$ , and it comes, for all  $k \geq n_\alpha$ ,  $\lambda_i \gamma_k \leq C\gamma_k < 1$ . Then, with the help of the Taylor's expansion of the functional  $x \mapsto \ln(1 - x)$ , one can check that for all  $i \in I$  and for all  $k \geq n_\alpha$ ,

$$-\lambda_i \gamma_k \geq \ln(1 - \lambda_i \gamma_k) \geq -\lambda_i \gamma_k - \frac{\lambda_i^2 \gamma_k^2}{1 - C\gamma_{n_\alpha}} = -\lambda_i \gamma_k - c\gamma_k^2,$$

with  $c := \frac{1}{1 - C\gamma_{n_\alpha}}$ . Then, for all  $n, k \geq n_\alpha$ ,

$$\exp \left( - \sum_{j=k+1}^n ((\lambda_i + \lambda_{i'}) \gamma_j + 2c\gamma_j^2) \right) \leq c_{n,k} \leq \exp \left( - \sum_{j=k+1}^n (\lambda_i + \lambda_{i'}) \gamma_j \right).$$



With the help of an integral test for convergence,

$$\begin{aligned} c_{n,k} &\geq \exp \left( -(\lambda_i + \lambda_{i'}) \gamma_{k+1} - c_\gamma \int_{k+1}^n (\lambda_i + \lambda_{i'}) t^{-\alpha} dt - 2c\gamma_{k+1}^2 - 2cc_\gamma^2 \int_{k+1}^n t^{-2\alpha} dt \right) \\ c_{n,k} &\leq \exp \left( -(\lambda_i + \lambda_{i'}) \gamma_{k+1} - c_\gamma \int_{k+1}^n (\lambda_i + \lambda_{i'}) t^{-\alpha} dt \right). \end{aligned}$$

Then,

$$\begin{aligned} c_{n,k} &\geq \exp \left( -(\lambda_i + \lambda_{i'}) \left( \frac{c_\gamma}{1-\alpha} \left( (k+1)^{1-\alpha} - n^{1-\alpha} \right) - \gamma_{k+1} \right) \right) \\ &\quad \times \exp \left( 2c \left( \gamma_{k+1}^2 - \frac{c_\gamma^2}{1-2\alpha} \left( n^{1-2\alpha} - (k+1)^{1-2\alpha} \right) \right) \right) \\ c_{n,k} &\leq \exp \left( -(\lambda_i + \lambda_{i'}) \left( \frac{c_\gamma}{1-\alpha} \left( (k+1)^{1-\alpha} - n^{1-\alpha} \right) - \gamma_{k+1} \right) \right) \end{aligned}$$

We now give an upper bound of  $\sum_{k=1}^n \gamma_k^2 c_{n,k}$ . Since  $0 < \lambda_{\min} \leq \lambda_i \leq C$  for all  $i \in I$ , there is a rank  $n_\alpha$ , only depending on  $\lambda_{\min}, C, c_\gamma$  and  $\alpha$ , such that the functional  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  defined for all  $t \in \mathbb{R}$  by

$$\varphi(t) := c_\gamma^2 t^{-2\alpha} \exp \left( -(\lambda_i + \lambda_{i'}) \left( \frac{c_\gamma}{1-\alpha} \left( (t+1)^{1-\alpha} - n^{1-\alpha} \right) - c_\gamma (t+1)^{-\alpha} \right) \right),$$

is increasing on  $[n_\alpha, +\infty]$ . For the sake of simplicity, let us consider that  $n_\alpha = 0$ . Then, with the help of an integral test for convergence,

$$\begin{aligned} \sum_{k=1}^n \gamma_k^2 c_{n,k} &\leq \int_0^n c_\gamma^2 t^{-2\alpha} \exp \left( -(\lambda_i + \lambda_{i'}) \left( \frac{c_\gamma}{1-\alpha} \left( (t+1)^{1-\alpha} - n^{1-\alpha} \right) - c_\gamma (t+1)^{-\alpha} \right) \right) dt \\ &= c_\gamma \frac{1}{\lambda_i + \lambda_{i'}} \left[ \exp \left( -(\lambda_i + \lambda_{i'}) \frac{c_\gamma}{1-\alpha} \left( (t+1)^{1-\alpha} - n^{1-\alpha} \right) \right) t^{-\alpha} \exp \left( -(\lambda_i + \lambda_{i'}) c_\gamma (t+1)^{-\alpha} \right) \right]_0^n \\ &\quad + c_\gamma \frac{1}{\lambda_i + \lambda_{i'}} \int_0^n e^{-(\lambda_i + \lambda_{i'}) \frac{c_\gamma}{1-\alpha} \left( (t+1)^{1-\alpha} - n^{1-\alpha} \right)} \left( t^{-1-\alpha} - (\lambda_i + \lambda_{i'}) c_\gamma t^{-2\alpha} \right) e^{-(\lambda_i + \lambda_{i'}) c_\gamma (t+1)^{-\alpha}} dt \end{aligned} \quad (36)$$

Then, since for all  $i \in I$ ,  $0 < \lambda_{\min} \leq \lambda_i \leq C$ , one can check that there is a positive sequence  $(\epsilon_n)_{n \geq 1}$  only depending on  $\alpha, c_\gamma, \lambda_{\min}, C$  such that

$$\sum_{k=1}^n \gamma_k^2 c_{n,k} \leq \frac{\gamma_n}{\lambda_i + \lambda_{i'}} + \epsilon_n \gamma_n, \quad \text{and} \quad \lim_{n \rightarrow \infty} \epsilon_n = 0.$$

With analogous calculus, one can check that there is a positive sequence  $(\epsilon'_n)_{n \geq 1}$  only depending on  $\alpha, c_\gamma, \lambda_{\min}, C$  such that

$$\sum_{k=1}^n \gamma_k^2 c_{n,k} \geq \frac{\gamma_n}{\lambda_i + \lambda_{i'}} - \epsilon'_n \gamma_n, \quad \text{and} \quad \lim_{n \rightarrow \infty} \epsilon'_n = 0,$$

which concludes the proof.

### A.3 Proof of Lemma 6.3

We only give the bound of the quadratic mean error since the almost sure rate of convergence is quite straightforward. First, since

$$\begin{aligned} (m_j - \bar{m}_j) \otimes (m_j - \bar{m}_j) - (m_j - m) \otimes (m_j - m) \\ = (m_j - m + m - \bar{m}_j) \otimes (m_j - m + m - \bar{m}_j) - (m_j - m) \otimes (m_j - m) \\ = (m - \bar{m}_j) \otimes (m_j - m) + (m_j - m) \otimes (m - \bar{m}_j) + (m - \bar{m}_j) \otimes (m - \bar{m}_j), \end{aligned}$$

and by linearity, let

$$\begin{aligned} (\star) &:= \Sigma_n - \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \\ &= -\frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \\ &\quad - \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \\ &\quad + \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right). \end{aligned}$$

Then, we have to bound the three terms on the right-hand side of previous equality.

$$\text{Bounding } \mathbb{E} \left[ \left\| \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \right\|_F^2 \right].$$

First, applying Lemma 6.9 and equality (2), let

$$\begin{aligned} (*) &:= \mathbb{E} \left[ \left\| \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \right\|_F^2 \right] \\ &\leq \left( \frac{1-\delta}{n^{1-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \sqrt{\mathbb{E} \left[ \left\| \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \right\|_F^2 \right]} \right)^2 \\ &\leq \left( \frac{1-\delta}{n^{1-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp\left(-\frac{k^{1-s}}{1-s}\right) \sqrt{\mathbb{E} \left[ \left\| \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right\|^2 \left\| \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right\|^2 \right]} \right)^2. \end{aligned}$$

Applying Cauchy-Schwarz's inequality,

$$(*) \leq \left( \frac{1-\delta}{n^{1-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^{\delta+s}} e^{-\frac{k^{1-s}}{1-s}} \left( \mathbb{E} \left[ \left\| \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right\|^4 \right] \right)^{\frac{1}{4}} \left( \mathbb{E} \left[ \left\| \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right\|^4 \right] \right)^{\frac{1}{4}} \right)^2.$$

First, note that thanks to Lemma 6.1

$$\mathbb{E} \left[ \left\| \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right\|^4 \right] = O \left( \exp \left( \frac{2k^{1-s}}{1-s} \right) k^{2s} \right).$$

Furthermore, applying Lemmas 6.9 and Lemma 6.7 as well as inequality (9),

$$\begin{aligned} \mathbb{E} \left[ \left\| \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right\|^4 \right] &\leq \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} \left( \mathbb{E} \left[ \|\bar{m}_j - m\|^4 \right] \right)^{\frac{1}{4}} \right)^4 \\ &\leq C_2' \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} \frac{1}{j^{1/2}} \right)^4 \\ &= O \left( \exp \left( 2 \frac{k^{1-s}}{(1-s)} \right) k^{4s} k^{-2} \right). \end{aligned} \quad (37)$$

Then, applying Lemma 6.7,

$$(*) = O \left( \left( \frac{1-\delta}{n^{1-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^{\delta+1/2-s/2}} \right)^2 \right) = O \left( \frac{1}{n^{1-s}} \right).$$

With analogous calculus, one can check that

$$\mathbb{E} \left[ \left\| \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} e^{-\frac{k^{1-s}}{1-s}} \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \right\|_F^2 \right] = O \left( \frac{1}{n^{1-s}} \right).$$

$$\text{Bounding } \mathbb{E} \left[ \left\| \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp \left( -\frac{k^{1-s}}{1-s} \right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \right\|_F^2 \right].$$

First, applying Lemma 6.9 and equality (2), let

$$\begin{aligned} (**) &= \mathbb{E} \left[ \left\| \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp \left( -\frac{k^{1-s}}{1-s} \right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \right\|_F^2 \right] \\ &\leq \left( \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} e^{-\frac{k^{1-s}}{1-s}} \sqrt{\mathbb{E} \left[ \left\| \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right) \right\|_F^2 \right]} \right)^2 \\ &= \left( \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} e^{-\frac{k^{1-s}}{1-s}} \sqrt{\mathbb{E} \left[ \left\| \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (\bar{m}_j - m) \right\|_F^4 \right]} \right)^2. \end{aligned}$$

Then, applying inequality (37) and Corollary 6.2,

$$(**) = O \left( \left( \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{1+\delta-s}} \right)^2 \right) = O \left( \frac{1}{n^{2(1-s)}} \right),$$

which concludes the proof.

#### A.4 Proof of Lemma 6.4

We just give the proof for the rate of convergence in quadratic mean, the proof of the almost sure rate of convergence is quite straightforward. Let

$$\begin{aligned} (\star) &:= \frac{1-\delta}{n^{1-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp \left( -\frac{k^{1-s}}{1-s} \right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) - \bar{\Sigma}_n \\ &= \left( \frac{1-\delta}{n^{1-\delta}} - \frac{1}{\sum_{k=1}^n k^{-\delta}} \right) \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp \left( -\frac{k^{1-s}}{1-s} \right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \\ &\quad + \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta}} \left( k^{-s} \exp \left( -\frac{k^{1-s}}{1-s} \right) - b_k^{-1} \right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \end{aligned}$$

We now bound the quadratic mean of each term on the right-hand side of previous equality.

First, note that with the help of an integral test for convergence,

$$\frac{1}{1-\delta} \left( (n+1)^{1-\delta} \right) = \int_0^{n+1} t^{-\delta} dt \geq \sum_{k=1}^n k^{-\delta} \geq \int_1^n t^{-\delta} dt = \frac{1}{1-\delta} (n^{1-\delta} - 1).$$

Then,

$$\begin{aligned} \left| \frac{1-\delta}{n^{1-\delta}} - \frac{1}{\sum_{k=1}^n k^{-\delta}} \right| &= \left| \frac{\frac{n^{1-\delta}}{1-\delta} - \sum_{k=1}^n k^{-\delta}}{\frac{n^{1-\delta}}{1-\delta} \sum_{k=1}^n k^{-\delta}} \right| \\ &\leq (1-\delta) \frac{(n+1)^{1-\delta} - n^{1-\delta} + 1}{n^{1-\delta} (n^{1-\delta} - 1)} \\ &= O \left( \frac{1}{n^{2-2\delta}} \right). \end{aligned}$$

Then, applying Lemma 6.9, there is a positive constant  $C$  such that for all  $n \geq 1$ ,

$$\begin{aligned} u_n &:= \mathbb{E} \left[ \left\| \left( \frac{1-\delta}{n^{1-\delta}} - \frac{1}{\sum_{k=1}^n k^{-\delta}} \right) \sum_{k=1}^n \frac{1}{k^{\delta+s}} e^{-\frac{k^{1-s}}{1-s}} \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \right\|_F^2 \right] \\ &\leq \frac{C}{n^{4-4\delta}} \left( \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp \left( -\frac{k^{1-s}}{1-s} \right) \sqrt{\mathbb{E} \left\| \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \right\|_F^2} \right)^2. \end{aligned}$$

Furthermore, applying equality (2)

$$u_n \leq \frac{C}{n^{4-4\delta}} \left( \sum_{k=1}^n \frac{1}{k^{\delta+s}} \exp \left( -\frac{k^{1-s}}{1-s} \right) \sqrt{\mathbb{E} \left\| \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right\|_F^4} \right)^2$$

Finally, applying Lemma 6.1 and since  $\delta < (1+s)/2$ ,

$$u_n = O \left( \frac{C}{n^{4-4\delta}} \left( \sum_{k=1}^n \frac{1}{k^{\delta}} \right)^2 \right) = O \left( \frac{1}{n^{2-2\delta}} \right) = o \left( \frac{1}{n^{1-s}} \right).$$

In the same way, with the help of an integral test for convergence,

$$\begin{aligned} b_n &:= \sum_{k=1}^n \exp \left( \frac{k^{1-s}}{1-s} \right) \leq \int_0^n \exp \left( \frac{t^{1-s}}{(1-s)} \right) dt \\ &\leq n^s \exp \left( \frac{n^{1-s}}{(1-s)} \right) + s \int_0^n \exp \left( \frac{t^{1-s}}{(1-s)} \right) t^{s-1} dt \\ &= n^s \exp \left( \frac{n^{1-s}}{1-s} \right) + s n^{2s-1} \exp \left( \frac{n^{1-s}}{1-s} \right) + o \left( n^{2s-1} \exp \left( \frac{n^{1-s}}{1-s} \right) \right). \end{aligned}$$

Thus, one can check that there is a positive constant  $c$  such that for all  $n \geq 1$ ,

$$b_n \geq n^s \exp \left( \frac{n^{1-s}}{1-s} \right) + c n^{2s-1} \exp \left( \frac{n^{1-s}}{1-s} \right)$$

Then,

$$\begin{aligned} \left| \frac{1}{b_n} - n^{-s} \exp \left( -\frac{n^{1-s}}{1-s} \right) \right| &= \frac{n^{-s} \exp \left( -\frac{n^{1-s}}{1-s} \right)}{b_n} \left| b_n - n^s \exp \left( \frac{n^{1-s}}{1-s} \right) \right| \\ &= O \left( n^{-1} \exp \left( -\frac{n^{1-s}}{1-s} \right) \right) \end{aligned}$$

Thus, applying Lemma 6.9, there is a positive constant  $C$  such that for all  $n \geq 1$ ,

$$\begin{aligned} v_n &:= \mathbb{E} \left[ \left\| \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^{\delta}} \left( k^{-s} e^{-\frac{k^{1-s}}{1-s}} - b_k^{-1} \right) \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \right\|_F^2 \right] \\ &\leq \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^{\delta}} k^{-1} e^{-\frac{k^{1-s}}{1-s}} \sqrt{\mathbb{E} \left[ \left\| \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \otimes \left( \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right) \right\|_F^2 \right]} \right)^2. \end{aligned}$$

Finally, applying equality (2) and Lemma 6.1,

$$\begin{aligned}
v_n &\leq \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta} k^{-1} \exp \left( -\frac{k^{1-s}}{1-s} \right) \sqrt{\mathbb{E} \left[ \left\| \sum_{j=1}^k e^{\frac{j^{1-s}}{2(1-s)}} (m_j - m) \right\|_F^4 \right]} \right)^2 \\
&= O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^{\delta+1-s}} \right)^2 \right) \\
&= O \left( \frac{1}{n^{2(1-s)}} \right),
\end{aligned}$$

which concludes the proof.

### A.5 Proof of Lemma 6.5

*Proof of Lemma 6.5.* This proof is a direct application of Lemma 6.1. In order to convince the reader, we just give one proof, and the other ones are analogous. Applying Lemma 6.9 and 6.1 as well as Corollary 6.2,

$$\begin{aligned}
\mathbb{E} \left[ \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \|A_{1,k}\|^2 \right)^2 \right] &\leq \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta b_k} \sqrt{\mathbb{E} [\|A_{1,k}\|^4]} \right)^2 \\
&= O \left( \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \right)^2 \left( \sum_{k=1}^n \frac{1}{k^\delta} k^{\alpha-s} \right)^2 \right) \\
&= O \left( \frac{1}{n^{2(s-\alpha)}} \right),
\end{aligned}$$

which concludes the proof. □

We now give the "almost sure version" of Lemma 6.5.

**Lemma A.1.** *Suppose Assumptions (A1) to (A5a') hold. Then, for all  $i, j \in \{1, 2\}$ , and for all  $\gamma > 0$ ,*

$$\begin{aligned}
\mathbb{E} \left[ \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \|A_{i,k}\| \|A_{j,k}\| \right)^2 \right] &= o \left( \frac{(\ln n)^\gamma}{n^{1-s}} \right), \\
\mathbb{E} \left[ \left( \frac{1}{\sum_{k=1}^n k^{-\delta}} \sum_{k=1}^n \frac{1}{k^\delta b_k} \|A_{i,k}\| \|M_{k+1}\| \right)^2 \right] &= o \left( \frac{(\ln n)^\gamma}{n^{1-s}} \right).
\end{aligned}$$

The proof is not given since it is quite closed to the one of Lemma 6.5.

## A.6 Proof of Lemma 5.1

*Proof of Lemma 5.1.* First, for all  $h \in H$ , let us define the function  $\varphi_h : [0, 1] \rightarrow \mathcal{S}(H)$ , defined for all  $t \in [0, 1]$  by

$$\begin{aligned}\varphi_h(t) &= \mathbb{E} [\nabla_h g(X, m^v + t(h - m^v)) \otimes \nabla_h g(X, m^v + t(h - m^v))] \\ &= \mathbb{E} \left[ \frac{X - m^v + t(h - m^v)}{\|X - m^v + t(h - m^v)\|} \otimes \frac{X - m^v + t(h - m^v)}{\|X - m^v + t(h - m^v)\|} \right].\end{aligned}$$

In what follows, we will denote  $A(t) := X - m^v + t(h - m^v)$ . Note that

$$\varphi_h(0) = \mathbb{E} [\nabla_h g_v(X, m) \otimes \nabla_h g_v(X, m)] \quad \varphi_h(1) = \mathbb{E} [\nabla_h g_v(X, h) \otimes \nabla_h g_v(X, h)]$$

and that the functional  $\varphi_h$  is differentiable, and its derivative is defined for all  $t \in [0, 1]$  by

$$\begin{aligned}\varphi'_h(t) &= \mathbb{E} \left[ \frac{1}{\|A(t)\|^4} \langle h - m^v, A(t) \rangle A(t) \otimes A(t) \right] + \mathbb{E} \left[ \frac{1}{\|A(t)\|^2} (h - m^v) \otimes A(t) \right] \\ &\quad + \mathbb{E} \left[ \frac{1}{\|A(t)\|^2} A(t) \otimes (h - m^v) \right].\end{aligned}$$

Then, applying Cauchy-Schwarz's inequality,

$$\|\varphi'_h(t)\|_F \leq 3\mathbb{E} \left[ \frac{1}{\|A(t)\|} \right] \|m^v - h\|.$$

Thus, let  $\epsilon > 0$ , thanks to Assumption **(H2)**, there is a positive constant  $C_{\|m^v\|+\epsilon}$  such that for all  $t \in [0, 1]$  and for all  $h \in \mathcal{B}(m^v, \epsilon)$ ,

$$\|\varphi'_h(t)\|_F \leq C_{\|m^v\|+\epsilon} \|m^v - h\|.$$

Finally,

$$\begin{aligned}\|\mathbb{E} [\nabla_h g_v(X, m) \otimes \nabla_h g_v(X, m)] - \mathbb{E} [\nabla_h g_v(X, h) \otimes \nabla_h g_v(X, h)]\|_F &= \|\varphi_h(1) - \varphi_h(0)\|_F \\ &= \left\| \int_0^1 \varphi'_h(t) dt \right\|_F \\ &\leq C_{\|m^v\|+\epsilon} \|m^v - h\|,\end{aligned}$$

which concludes the proof.  $\square$

## References

- Bach, F. (2014). Adaptivity of averaged stochastic gradient descent to local strong convexity for logistic regression. *The Journal of Machine Learning Research*, 15(1):595–627.
- Bach, F. and Moulines, E. (2013). Non-strongly-convex smooth stochastic approximation

- with convergence rate  $O(1/n)$ . In *Advances in Neural Information Processing Systems*, pages 773–781.
- Boyd, S. and Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.
- Cardot, H., Cénac, P., and Godichon-Baggioni, A. (2015). Online estimation of the geometric median in Hilbert spaces: non asymptotic confidence balls. Technical report, [arXiv:1501.06930](https://arxiv.org/abs/1501.06930).
- Cardot, H., Cénac, P., and Zitt, P.-A. (2013). Efficient and fast estimation of the geometric median in Hilbert spaces with an averaged stochastic gradient algorithm. *Bernoulli*, 19(1):18–43.
- Cardot, H. and Godichon-Baggioni, A. (2015). Fast estimation of the median covariation matrix with application to online robust principal components analysis. *TEST*, pages 1–20.
- Chakraborty, A. and Chaudhuri, P. (2014). The spatial distribution in infinite dimensional spaces and related quantiles and depths. *The Annals of Statistics*, 42:1203–1231.
- Chaudhuri, P. (1996). On a geometric notion of quantiles for multivariate data. *J. Amer. Statist. Assoc.*, 91(434):862–872.
- Delyon, B. and Juditsky, A. (1992). Stochastic optimization with averaging of trajectories. *Stochastics: An International Journal of Probability and Stochastic Processes*, 39(2-3):107–118.
- Delyon, B. and Juditsky, A. (1993). Accelerated stochastic approximation. *SIAM Journal on Optimization*, 3(4):868–881.
- Dippon, J. and Renz, J. (1997). Weighted means in stochastic approximation of minima. *SIAM Journal on Control and Optimization*, 35(5):1811–1827.
- Duflo, M. (1996). *Algorithmes stochastiques*. Springer Berlin.
- Duflo, M. (1997). *Random iterative models*, volume 34 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin. Translated from the 1990 French original by Stephen S. Wilson and revised by the author.
- Gahbiche, M. and Pelletier, M. (2000). On the estimation of the asymptotic covariance matrix for the averaged robbins–monro algorithm. *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, 331(3):255–260.
- Gervini, D. (2008). Robust functional estimation using the median and spherical principal components. *Biometrika*, 95(3):587–600.
- Godichon-Baggioni, A. (2016a). Estimating the geometric median in hilbert spaces with stochastic gradient algorithms:  $L_p$  and almost sure rates of convergence. *Journal of Multivariate Analysis*, 146:209–222.



- Godichon-Baggioni, A. (2016b). Lp and almost sure rates of convergence of averaged stochastic gradient algorithms with applications to online robust estimation. *arXiv preprint [arXiv:1609.05479](#)*.
- Godichon-Baggioni, A. and Portier, B. (2016). An averaged projected robbins-monro algorithm for estimating the parameters of a truncated spherical distribution. *arXiv preprint [arXiv:1606.04276](#)*.
- Haldane, J. B. S. (1948). Note on the median of a multivariate distribution. *Biometrika*, 35(3-4):414–417.
- Hallin, M. and Paindaveine, D. (2006). Semiparametrically efficient rank-based inference for shape. i. optimal rank-based tests for sphericity. *The Annals of Statistics*, 34(6):2707–2756.
- Huber, P. and Ronchetti, E. (2009). *Robust Statistics*. John Wiley and Sons, second edition.
- Jakubowski, A. (1988). Tightness criteria for random measures with application to the principle of conditioning in Hilbert spaces. *Probab. Math. Statist.*, 9(1):95–114.
- Juditsky, A., Nesterov, Y., et al. (2014). Deterministic and stochastic primal-dual subgradient algorithms for uniformly convex minimization. *Stochastic Systems*, 4(1):44–80.
- Kemperman, J. (1987). The median of a finite measure on a Banach space. In *Statistical data analysis based on the  $L_1$ -norm and related methods (Neuchâtel, 1987)*, pages 217–230. North-Holland, Amsterdam.
- Kraus, D. and Panaretos, V. M. (2012). Dispersion operators and resistant second-order functional data analysis. *Biometrika*, 99:813–832.
- Kushner, H. J. and Yin, G. (2003). *Stochastic approximation and recursive algorithms and applications*, volume 35. Springer Science & Business Media.
- Maronna, R. A., Martin, R. D., and Yohai, V. J. (2006). *Robust statistics*. Wiley Series in Probability and Statistics. John Wiley & Sons, Ltd., Chichester. Theory and methods.
- Minsker, S., Srivastava, S., Lin, L., and Dunson, D. (2014). Scalable and robust bayesian inference via the median posterior. In *Proceedings of the 31st International Conference on Machine Learning (ICML-14)*, pages 1656–1664.
- Mokkadem, A. and Pelletier, M. (2006). Convergence rate and averaging of nonlinear two-time-scale stochastic approximation algorithms. *Ann. Appl. Probab.*, 16(3):1671–1702.
- Nemirovski, A., Juditsky, A., Lan, G., and Shapiro, A. (2009). Robust stochastic approximation approach to stochastic programming. *SIAM Journal on Optimization*, 19(4):1574–1609.
- Oja, H. and Niinimaa, A. (1985). Asymptotic properties of the generalized median in the case of multivariate normality. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 372–377.

- Pelletier, M. (1998). On the almost sure asymptotic behaviour of stochastic algorithms. *Stochastic processes and their applications*, 78(2):217–244.
- Pelletier, M. (2000). Asymptotic almost sure efficiency of averaged stochastic algorithms. *SIAM J. Control Optim.*, 39(1):49–72.
- Polyak, B. and Juditsky, A. (1992). Acceleration of stochastic approximation. *SIAM J. Control and Optimization*, 30:838–855.
- Robbins, H. and Monro, S. (1951). A stochastic approximation method. *The annals of mathematical statistics*, pages 400–407.
- Ruppert, D. (1988). Efficient estimations from a slowly convergent robbins-monro process. Technical report, Cornell University Operations Research and Industrial Engineering.
- Serfling, R. (2006). Depth functions in nonparametric multivariate inference. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, 72:1.
- Vardi, Y. and Zhang, C.-H. (2000). The multivariate  $L_1$ -median and associated data depth. *Proc. Natl. Acad. Sci. USA*, 97(4):1423–1426.
- Weng, J., Zhang, Y., and Hwang, W.-S. (2003). Candid covariance-free incremental principal component analysis. *IEEE Trans. Pattern Anal. Mach. Intell.*, 25:1034–1040.